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# On Lévy Processes for Option Pricing :

## Numerical Methods and Calibration to Index Options

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À mes parents.

# Introduzione e Risultati

Dal famoso articolo di Black and Scholes del 1973 sul prezzaggio delle opzioni, una notevole quantità di articoli è stata scritta sull'argomento. Tuttavia, col passare del tempo, l'ipotesi fondamentale del modello di Black-Scholes che il sottostante segua una diffusione lognormale con una volatilità costante si è rivelata sempre più difficile da sostenere. Di conseguenza, negli ultimi 30 anni, tanti modelli sono stati presentati come alternativa a questo modello.
Infatti, dopo il crollo finanziario del 19 ottobre 1987, si è iniziato ad osservare divari molto significativi tra prezzi di opzioni su vari indici azionari e prezzi dati dal modello di Black-Scholes. In effetti, da allora, per ricavare il prezzo di mercato di diversi opzioni call e put con il modello di Black-Scholes, occorre utilizzare volatilità diverse per ogni prezzo di esercizio e scadenza (mentre il modello di Black-Scholes richiedeva un'unica volatilità ricavata dalla varianza storica del attivo sottostante). Queste volatilità vengono chiamate volatilità implicite. Questa peculiarità suggerisce che la distribuzione percepita dagli attori del mercato ed implicitamente incorporata nel prezzo delle opzioni stesse sia asimmetrica e deformata negativamente (cioè leptocurtica con una coda verso i valori negativi), in contrasto con la distribuzione simmetrica e leggermente positiva che caratterizza il modello di Black-Scholes. La figura formata

dalla volatilità implicita come funzione del prezzo di esercizio viene chiamata "smile" o "skew" della volatilità ed è generata dal fatto che la volatilità implicita di opzioni call in-the-money è parecchio superiore a quella di opzioni call out-of-the-money. Di solito, la pendenza dello smile decresce all'aumentare della scadenza. La presenza dello smile è generalmente attribuita ai timori da parte del mercato di futuri deprezzamenti significativi delle quotazioni. La ricerca di nuovi modelli capaci di integrare l'effetto smile è stata uno dei principali temi di ricerca della finanza quantitativa moderna.

Due ipotesi sono fondamentali nel modello di Black-Scholes per poter prezzare derivati: i) i rendimenti del sottostante devono dipendere da un'unica fonte di incertezza, ii) i prezzi devono seguire una traiettoria continua (un moto Browniano per essere precisi). Sotto tale ipotesi, può essere costruito un portafoglio che assicuri in ogni momento una copertura perfetta della posizione in opzioni, determinando così un prezzo unico per l'opzione.

Di conseguenza, tutte le estensioni del modello di Black-Scholes che tentano di catturare l'effetto smile della volatilità si basano sul rilassamento di almeno una delle ipotesi suddette. Rimuovendo l'ipotesi di un'unica fonte di incertezza si arriva ai modelli di volatilità stocastica, in cui il parametro della volatilità segue una diffusione correlata con quella del sottostante. Uno di questi modelli è stato presentato da Heston [58]. Rimuovendo l'ipotesi di continuità della traiettoria si arriva invece ai modelli con salti, nei quali il prezzo del sottostante segue un processo di Lévy del tipo "jump-diffusion" (in cui l'evoluzione dei prezzi segue una diffusione punteggiata di salti ad intervalli casuali) o del tipo "pure jumps". I modelli con i salti attribuiscono gli errori del modello Black-Scholes ai timori di un futuro crollo delle quotazioni che non vengono presi

in conto nel prezzaggio delle opzioni. Questi modelli considerano dunque un crollo delle quotazioni come un evento plausibile. In effetti, se uno guarda un grafico di una serie storica dei prezzi di un indice azionario, si accorge subito della presenza di salti nei prezzi come il famoso gap inter giornaliero che è una prova evidente che i prezzi non seguono una diffusione ma effettuano salti.
Questa tesi tratta lo studio dei processi di Lévy per il prezzaggio di opzioni. I processi di Lévy sono un tema di ricerca molto studiato in finanza, e tanti modelli sono stati presentati durante gli ultimi dieci anni. In questa tesi, descriviamo i principali modelli di Lévy presentati nella letteratura scientifica. Focalizziamo la nostra attenzione su quattro modelli particolarmente famosi, due del tipo jump-diffusion (Merton normal jump-diffusion e Kou double-exponential jump-diffusion) e due modelli "pure jumps" (Variance Gamma e Normal Inverse Gaussian) dei quali esponiamo le principali proprietà matematiche e tecniche per la loro simulazione.

Nel primo capitolo, riassumiamo gli strumenti matematici utili in finanza quantitativa e più particolarmente nel prezzaggio di opzioni. Presentiamo le caratteristiche dei processi stocastici e il loro uso per il prezzaggio di opzioni.
Nel secondo capitolo, descriviamo le debolezze del modello di Black-Scholes e dimostriamo da un punto di vista statistico quanto l'ipotesi di rendimenti lognormali, sottintesa da questo modello, sia errata per rappresentare i rendimenti di mercato. Infine, presentiamo qualche modello alternativo.
Nel terzo capitolo, introduciamo i processi di Lévy ed esponiamo le loro principali proprietà matematiche, iniziando con il processo di Poisson, punto di partenza della costruzione di processi con i salti.

Nel quarto capitolo, presentiamo i quattro modelli di Lévy che abbiamo utilizzato per prezzare opzioni: Merton normal jump-diffusion, Kou double-exponential jump-diffusion, Variance Gamma e Normal Inverse Gaussian. Una parte importante del capitolo è dedicata alla simulazione di questi processi. In una parte finale, diamo le funzioni caratteristiche neutrali al rischio di ogni modello, dato che ne avremo bisogno nel capitolo successivo sul prezzaggio.
Il quinto capitolo tratta alcuni metodi per il prezzaggio di opzioni. I modelli con i salti introducono nuove fonti di rischio, incluso nel prezzo di un opzione, che non sono prezzate da nessuno strumento finanziario quotato su un mercato. Ne risulta che la metodologia di prezzaggio utilizzata nel modello Black-Scholes (basata sulla costruzione di un portafoglio di replicazione secondo l'ipotesi di assenza di arbitraggio) non funziona. Tuttavia, basandosi sulla formula di prezzaggio neutrale al rischio, visto che la funzione caratteristica di un processo di Lévy è sempre determinabile (anche se la funzione di densità non lo è), possiamo utilizzare metodi di prezzaggio basati sulla trasformata di Fourier. Spieghiamo in una prima parte il metodo di prezzaggio di opzioni con la FFT, sviluppata da Carr e Madan [24]. In una seconda parte, estendiamo il discorso alla Trasformata di Fourier Frazionale (FRFT), algoritmo molto più veloce della semplice FFT. Infine, diamo risultati sulla bontà dell'approssimazione dei prezzi delle opzioni effettuata con i precedenti algoritmi.
Nel sesto e ultimo capitolo ci occupiamo della calibrazione dei parametri sui prezzi di opzioni rilevati sul mercato. Per cominciare, selezioniamo quattro gruppi di opzioni e mostriamo l'errore ottenuto con il modello di Black-Scholes nell'approssimazione dei veri prezzi. In seguito, presentiamo il metodo dei minimi quadrati non-lineari usato per calibrare i parametri dei vari processi di Lévy

sui prezzi di mercato. Una volta ottenuti i parametri dei modelli, calcoliamo i nuovi prezzi che risultano essere decisamente più vicini ai prezzi di mercato di quanto lo siano i prezzi ottenuti con il modello di Black-Scholes.

In particolare, con i diversi modelli di Lévy analizzati, risulta possibile calibrare con alta precisione i prezzi di un gruppo di opzioni con un'unica scadenza. Tuttavia, risulta molto più difficile calibrare accuratamente il modello su un intero gruppo di opzioni (su indice azionario) con diverse scadenze. Anche se proviamo ad utilizzare diversi valori iniziali dei parametri, l'algoritmo difficilmente riesce a raggiungere un minimo soddisfacente e in media, otteniamo un errore dal 0.5% al 1.5% per opzione (l'errore è ovviamente più alto per valori estremi del prezzo di esercizio). La struttura dei processi di Lévy sembra non essere sufficientemente flessibile per riprodurre accuratamente e completamente la struttura a termine della volatilità implicita. Un metodo molto più efficiente potrebbe essere l'utilizzo di modelli che includono sia la volatilità stocastica sia i salti. Lo studio di questi modelli ibridi, e in particolare dei metodi utilizzati per il prezzaggio e la calibrazione, potrebbe rappresentare una giusta estensione di questo lavoro.
In questa tesi, non abbiamo potuto parlare di due argomenti fondamentali quali il prezzaggio di opzioni esotiche e l'hedging, cioè la copertura del rischio derivante da una posizione in opzioni. Il primo di questi due è uno dei maggiori obiettivi dei modelli di prezzaggio di opzioni, particolarmente quelli basati sui processi di Lévy. In effetti, lo scopo della calibrazione del modello è ottenere una stima dei parametri sottointesi nei prezzi di opzioni vanilla (quotati su un mercato) in modo da poter prezzare prodotti esotici OTC con payoff non-

vanilla (come le opzioni asiatiche o barriera). Molto spesso, il prezzaggio di opzioni esotiche con i modelli di Lévy viene fatto tramite metodi Monte-Carlo, anche se esistono metodi numerici finalizzati a risolvere l'equazione differenziale parziale associata al modello. D'altra parte, quando si ha a che fare con opzioni, e particolarmente opzioni esotiche, l'hedging è un tema importante almeno quanto il prezzaggio. Vista la rischiosità degli strumenti derivati, lo scopo di tanti modelli è giustamente quantificare e controllare questo rischio. Di conseguenza, il prezzaggio di opzioni esotiche e l'hedging potrebbero rappresentare due possibili estensioni basate sui risultati di questa tesi.

Il codice scritto per realizzare le simulazioni, il prezzaggio e la calibrazione nei capitoli 4, 5 e 6 è una parte fondamentale di questa tesi. Il codice è disponibile nell'appendice ma anche scaricabile dal seguente sito web:

*http://ddeville.110mb.com/thesis/*

Il codice è stato scritto con MATLAB. La principale ragione per la quale abbiamo scelto questo linguaggio di programmazione piuttosto che un altro come C o C++ è la semplicità. In effetti, il codice MATLAB è semplice da scrivere ma soprattuto facile da leggere e comprendere anche da una persona che ha solo qualche base di programmazione.

# Introduction

Since Black and Scholes published their article on option pricing in 1973, there has been an explosion of theoretical and empirical work on the subject. However, over the last thirty years, a vast number of pricing models have been proposed as an alternative to the classic Black-Scholes approach, whose assumption of lognormal stock diffusion with constant volatility is considered always more flawed.

One major reason is that since the stock market crash of October 19, 1987, deviations of stock index option prices from the benchmark Black-Scholes model have been extraordinarily pronounced. In fact, since then, to equate the Black-Scholes formula with quoted prices of European calls and puts, it is generally necessary to use different volatilities, so-called implied volatilities, for different option strikes and maturities (the Black-Scholes model required a constant volatility based on the subjacent historical volatility). That feature suggests that the distribution perceived by market participant and incorporated into option prices is substantially negatively skewed (that is to say leptokurtic with a fat tail on the negative side), in contrast to the essentially symmetric and slightly positively lognormal distribution underlying the Black-Scholes model. The pattern formed by the implied volatilities across the strikes is then called

volatility *smile* or *skew*, due to the fact that the implied volatility of in-the-money call options is pretty much higher than the one of out-of-the-money options. Typically, the steepness of the skew decreases with increasing option maturities. The existence of the skew is often attributed to fear of large downward market movements. The research of a new form of models able to incorporate the smile has been one of the most active fields of studies in modern quantitative finance.

There are two assumptions that have to be made in order to price derivatives with the Black-Scholes model: returns are subject to a single source of uncertainty and asset prices follow continuous sample paths (a Brownian motion). Then, under these two assumptions, a continuously rebalanced portfolio can be used to perfectly hedge an options position, thus determining a unique price for the option.

Therefore, extensions of the Black-Scholes model that capture the existence of volatility smile can, broadly speaking, be grouped in two approaches, each one relaxing one of these two assumptions. Relaxing the assumption of a unique source of uncertainty leads to the stochastic volatility family of models, where the volatility parameter follows a separate diffusion, as proposed by Heston [58]. Relaxing the assumption of continuous sample paths, leads to jump models, where stock prices follow an exponential Lévy process of jump-diffusion type (where evolution of prices is given by a diffusion process, punctuated by jumps at random intervals) or pure jumps type. Jump models attribute the biases in Black-Scholes model to fears of a further stock market crash. They would interpret the crash as a revelation that jumps can in fact occur. Looking to a plot of a stock index time series, there is clear evidence that prices don't

follow a diffusion process and actually jump.
This thesis deals with the study of Lévy processes for option pricing. Lévy processes are an active field of research in finance, and many models have been presented during the last decade. This thesis is not an attempt to describe all the Lévy models discussed in the literature or explain their mathematical properties. We focus on four famous models, two of jump-diffusion type (Merton normal jump-diffusion and Kou double-exponential jump-diffusion) and two pure jump models (Variance Gamma and Normal Inverse Gaussian) for which we describe their foremost mathematical characteristics and we concentrate on providing modeling tools.

In the first chapter, we present the mathematical tools useful for option pricing. We discuss some characteristics of stochastic processes and financial mathematics in continuous time. This chapter can be seen as a prerequisite.
In a second chapter, we discuss the limitations of the Black-Scholes model and describe its weaknesses. We also explain from a statistical point of view how the hypothesis of lognormal returns defined by the Black-Scholes model goes wrong in describing market returns. Finally, we roughly present some alternative models discussed in the literature.
In a third chapter, we introduce Lévy processes and present their major mathematical properties, beginning from Poisson process which is the starting point of jump processes.
In the fourth chapter, we present the four Lévy processes we selected for option pricing: Merton normal jump-diffusion, Kou double-exponential jump-diffusion, Variance Gamma and Normal Inverse Gaussian. An important part of the chapter is dedicated to the simulation of such processes. In the last part,

we give the neutral characteristic function of each model, needed for option pricing in the successive chapter.
The fifth chapter treats about option pricing methods. Jump models introduce forms of risk included in option prices that are not directly priced by any instrument currently traded in financial markets (unlike bonds for example). The result is that the Black-Scholes arbitrage-based methodology cannot be used. However, given the risk-neutral pricing formula and the fact that the characteristic function is always known for a Lévy process, even if the probability density is not, Fourier-based option pricing method are possible. We first expose the FFT option pricing method given by Carr and Madan [24], and extend it to the Fractional Fourier Transform (FRFT), a lot faster. We finally give some results about the goodness of the Fourier approximation of option prices.
Finally, the sixth chapter deals with the calibration of parameters to market option prices. We first select four sets of option market prices and show how bad results the Black-Scholes model gives. We then present the non-linear least-squares method used to recover the parameters of Lévy processes and finally show the improvement in the fitting of market and model prices with these models.

The code written to perform simulation, pricing and calibration in chapter 4, 5 and 6 is an important part of the thesis. All the code is available in the appendix at the end of the thesis and downloadable at the following website:

*http://ddeville.110mb.com/thesis/*

The code is written with MATLAB. The reason why we chose MATLAB rather

than a programming language like C or C++ is the level of difficulty. MATLAB code is easy to write and above all easy to read, even to someone with a very few knowledge of coding.

# Contents

**Introduzione e Risultati** **ix**

**Introduction** **xiv**

**1 Stochastic Processes and Mathematical Finance** **1**

1.1 Probability, Stochastic Processes, Filtrations . . . . . 1

1.2 Classes of Processes . . . . . 4

1.2.1 Markov Processes . . . . . 4

1.2.2 Martingales . . . . . 5

1.3 Characteristic Functions . . . . . 6

1.4 Brownian Motion . . . . . 7

1.4.1 Normal Distribution . . . . . 7

1.4.2 Brownian Motion . . . . . 7

1.5 Stochastic Integration - Itô Calculus . . . . . 11

1.5.1 Itô's Lemma . . . . . 11

1.5.2 Geometric Brownian Motion . . . . . 12

1.6 Mathematical Finance in Continuous Time . . . . . 15

1.6.1 Trading strategies . . . . . 15

1.6.2 Risk-Neutral Pricing Pricing and Martingale Measures . 16
1.6.3 Market Completeness . . . . . 20

**2 Black-Scholes Model and its Limitations 21**
2.1 The Black-Scholes Model . . . . . 21
2.2 The Limitations of the Black-Scholes Model . . . . . 26
2.2.1 The Evidence of the Volatility Smile . . . . . 26
2.2.2 The Incompleteness of Markets . . . . . 29
2.2.3 Are these Returns Really Log-normally Distributed? . . 30
2.2.4 New Models, New Solutions, New Problems . . . . . 32

**3 Lévy Processes 41**
3.1 Poisson Processes . . . . . 42
3.1.1 Poisson Process . . . . . 42
3.1.2 Compensated Poisson Process . . . . . 45
3.1.3 Compound Poisson Process . . . . . 45
3.2 Lévy Processes: Definition and properties . . . . . 48
3.2.1 Infinite Divisible Distributions . . . . . 50
3.2.2 The Lévy-Khintchine Formula . . . . . 51
3.2.3 The Lévy-Itô Decomposition . . . . . 53
3.2.4 The Lévy measure . . . . . 55
3.2.5 Lévy Processes as Martingales . . . . . 55
3.2.6 Path Properties . . . . . 56
3.3 Stochastic Calculus for Processes with Jumps . . . . . 57
3.3.1 Itô Formula for Jump processes . . . . . 57
3.3.2 Equivalence of Measures for Lévy Processes . . . . . 59

**4 Lévy Processes for Financial Modeling** **63**

4.1 Jump-Diffusion Models . . . . . 66

4.1.1 Merton Model . . . . . 68

4.1.2 Kou Model . . . . . 69

4.2 Infinite Activity Models . . . . . 70

4.2.1 Normal Inverse Gaussian . . . . . 71

4.2.2 Variance Gamma . . . . . 73

4.3 Simulation . . . . . 77

4.3.1 Jump Diffusion Processes . . . . . 77

4.3.2 Normal Inverse Gaussian Process . . . . . 83

4.3.3 Variance Gamma Process . . . . . 86

4.4 Risk-Neutral Characteristic Functions . . . . . 92

4.4.1 Merton Jump-Diffusion Model . . . . . 93

4.4.2 Kou Jump-Diffusion Model . . . . . 94

4.4.3 Normal Inverse Gaussian Model . . . . . 95

4.4.4 Variance Gamma Model . . . . . 95

**5 Option Pricing with FRFT** **101**

5.1 An Analytic Expression for the Fourier Transform . . . . . 102

5.1.1 Modified Option Price . . . . . 103

5.1.2 Time Value of an Option . . . . . 107

5.2 FFT . . . . . 109

5.3 FRFT . . . . . 112

5.4 Results . . . . . 114

5.4.1 The Dampening Factor $\alpha$ . . . . . 114

5.4.2 The Fourier Approximation . . . . . 116

5.4.3 The Implied Volatility Surface . . . . . 118

**6 Calibration** **121**

6.1 The Calibration Inputs . . . . . 122

6.1.1 The Risk-Free Interest Rate . . . . . 123

6.1.2 The Options Sets . . . . . 126

6.1.3 Dividends . . . . . 133

6.2 The Black-Scholes Results . . . . . 134

6.3 Estimation Methods: Non-Linear Least-Squares . . . . . 138

6.3.1 Gradient Descent . . . . . 139

6.4 Results . . . . . 140

**Conclusion** **149**

**A Matlab Code** **151**

**Bibliography** **177**

# Chapter 1

# Stochastic Processes and Mathematical Finance

## 1.1 Probability, Stochastic Processes, Filtrations

**Definition 1.1 (Algebra)** *Let $\Omega$ be a nonempty set, and let $\mathcal{F}$ be a collection of subsets of $\Omega$. We say that $\mathcal{F}$ is an algebra provided that:*

*(i)* $\Omega \in \mathcal{F}$ *and* $\emptyset \in \mathcal{F}$,

*(ii)* $A \in \mathcal{F} \Rightarrow A^c = \Omega \backslash A \in \mathcal{F}$,

*(iii)* $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

**Definition 1.2 ($\sigma$-algebra)** *An algebra $\mathcal{F}$ of subsets of $\Omega$ is called a $\sigma$-algebra on $\Omega$ if for any sequence $(\mathcal{A}_n)_{n\in\mathbb{N}} \in \mathcal{F}$, we have*

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$$

*Such a pair* $(\Omega, \mathcal{F})$ *is called a measurable space.*

Thus, a $\sigma$-algebra on $\Omega$ is a family of subsets of $\Omega$ closed under any countable collection of set operations. The $\sigma$-algebra generated by all open subsets is called the Borel $\sigma$-algebra: $\mathcal{B}(E)$.

**Definition 1.3 (Probability)** *Let* $\Omega$ *be a nonempty set, and let* $\mathcal{F}$ *be a* $\sigma$*-algebra of subsets of* $\Omega$*. A probability measure* $\mathbb{P}$ *is a function that, to every set* $A \in \mathcal{F}$ *assigns a number in* $[0, 1]$*, called the* probability *of* $A$ *and written* $\mathbb{P}(A)$*. We require:*

*(i)* $\mathbb{P}(\Omega) = 1$*, and*

*(ii) (countable additivity) whenever* $A_1, A_2, \ldots$ *is a sequence of disjoint sets in* $\mathcal{F}$*, then*

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n). \tag{1.1}$$

*The triple* $(\Omega, \mathcal{F}, \mathbb{P})$ *is called a* probability space*.*

A probability space is $\mathbb{P}$-complete if for each $B \subset A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$, we have $B \in \mathcal{F}$.

In a dynamic context, as time goes on, more information is progressively revealed to the observer. We must thus add some time-dependent ingredient to the structure of our probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Definition 1.4 (Filtration)** *A filtration (or information flow) on* $(\Omega, \mathcal{F}, \mathbb{P})$ *is an increasing family of* $\sigma$*-algebras* $(\mathcal{F}_t)_{t\in[0,T]}$*:*

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_T \subset \mathcal{F} \qquad \text{for} \qquad 0 \leq s < t \leq T.$$

$\mathcal{F}_t$ represents the information available at time $t$, and the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ represents the information flow evolving (increasing) with time.

A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration is called a *filtered probability space* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$.

**Definition 1.5 (Usual conditions)** *We say that a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ satisfies the "usual conditions" if:*

*(i)* *$\mathcal{F}$ is $\mathbb{P}$-complete.*

*(ii)* *$\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\Omega$. This means intuitively that we know which events are possible and which are not.*

*(iii)* *$(\mathcal{F}_t)_{t\in[0,T]}$ is right-continuous, i.e. $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$.*

**Definition 1.6 (Stochastic processes)** *A stochastic process $(X_t)_{t\in[0,T]}$ is a family of random variables indexed by time, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$.*

The time parameter $t$ may be either discrete or continuous. For each realization of the randomness $\omega$, the trajectory $X(\omega) : t \to X_t(\omega)$ defines a function of time called the *sample path* of the process. Thus stochastic processes can also be viewed as *random functions*.

**Definition 1.7 (Càdlàg function)** *A function $f : [0,T] \to \mathbb{R}^d$ is said to be* càdlàg *(from French "continu à droite, limite à gauche") if it is right-continuous with left limits. If the process is càglàd (left-continuous), one should be able to "predict" the value at $t$ -"see it coming"- knowing the values before $t$.*

**Definition 1.8 (Adapted processes)** *A stochastic process $(X_t)_{t\in[0,T]}$ is said to be* $\mathcal{F}_t$-adapted *(or nonanticipating with respect to the information structure $(\mathcal{F}_t)_{t\in[0,T]}$) if, for each $t \in [0,T]$, the value of $X_t$ is revealed at time $t$: the random variable $X_t$ is $\mathcal{F}_t$-measurable.*

**Definition 1.9 (Stopping times)** *A random time is a positive random variable $T \geq 0$ which represents the time at which some event is going to take place. If, given an information flow $\mathcal{F}_t$, someone can determine whether the event has happened ($\tau \leq t$) or not ($\tau > t$), the random time $\tau$ is called a* stopping time *(or nonanticipating random time). In other words, $\tau$ is a non-anticipating random time ($(\mathcal{F}_t)$-stopping time) if*

$$\forall t \geq 0, \qquad \{\tau \leq t\} \in \mathcal{F}_t.$$

## 1.2 Classes of Processes

### 1.2.1 Markov Processes

A *Markov process* is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The past history of the variable and the way that the present has emerged from the past are irrelevant (the past history is, say, integrated in the present value).

**Definition 1.10 (Markov process)** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $T$ be a fixed positive number, and let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration. Consider an adapted stochastic process $(X_t)_{t\in[0,T]}$. If, for a well-behaved (i.e. Borel-measurable) function $f$:*

$$\mathbf{E}[f(X_t)|\mathcal{F}_s] = \mathbf{E}[f(X_t)|X_s] \tag{1.2}$$

*the process* $(X_t)_{t\in[0,T]}$ *is a Markov process.*

### 1.2.2 Martingales

**Definition 1.11 (Martingale)** *A càdlàg stochastic process* $X = (X_t)_{t\in[0,T]}$ *is a martingale relative to* $(\mathbb{P}, \mathcal{F}_t)$ *if*

*(i)* $X$ *is* $\mathcal{F}_t$*-adapted,*

*(ii)* $\mathbf{E}[|X_t|] < \infty$ *for any* $t \in [0,T]$*,*

*(iii)* $\forall s < t$

$$\mathbf{E}[X_t|\mathcal{F}_s] = X_s \tag{1.3}$$

$X$ *is a supermartingale if in place of (iii)*

$$\mathbf{E}[X_t|\mathcal{F}_s] \leq X_s \qquad \forall s < t \tag{1.4}$$

$X$ *is a submartingale if in place of (iii)*

$$\mathbf{E}[X_t|\mathcal{F}_s] \geq X_s \qquad \forall s < t \tag{1.5}$$

In other words, the best prediction of a martingale's future value is its present value. Martingale have a useful interpretation in terms of dynamic games: a martingale is "constant on average", and models a fair game; a supermartingale is "decreasing on average", and models an unfavorable game; a submartingale is "increasing on average", and models a favorable game.

Martingales represent situations in which there is no drift, or tendency, though there may be a lot of randomness. In the typical statistical situation where we have *data = signal + noise*, martingales are used to model the noise component.

A familiar example of martingale is the Wiener process $W_t$.

## 1.3 Characteristic Functions

The characteristic function of a random variable is the Fourier transform of its distribution. Many probabilistic properties of random variables correspond to analytical properties of their characteristic functions, making this concept very useful for studying random variables.

**Definition 1.12 (Characteristic function)** *The characteristic function of the $\mathbb{R}^d$-valued random variable $X$ is the function $\Phi_X : \mathbb{R}^d \to \mathbb{R}$ defined by*

$$\Phi_X(t) = \mathbf{E}(e^{itX}) = \mathbf{E}cos(tX) + i\mathbf{E}sin(tX) \tag{1.6}$$

*Let $F_X$ be the distribution function of $X$. Then*

$$\Phi_X(t) = \mathbf{E}(e^{itX}) = \int_{-\infty}^{+\infty} e^{itx} dF(x) \tag{1.7}$$

*so that $\Phi$ is the Fourier transform of $F$, but without a constant multiplier such as $(2\pi)^{-1/2}$ which is used in much of Fourier analysis.*

The characteristic function of a random variable determines the probability distribution: two variables with the same characteristic function are identically distributed. A characteristic function is always continuous and verifies

$$\Phi_X(0) = 1 \qquad |\Phi_X(t)| \leq 1 \qquad \Phi_{aX+b}(t) = e^{itb}\Phi_X(at)$$

**Theorem 1.13** *If $\Phi_X$ is integrable, then $X$ has a density which is given by*

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \Phi_X(u) du.$$

**Example 1.14 (Gaussian CF)** *For a normal distribution N($\mu$,$\sigma^2$), we can define the density and characteristic function as:*

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \qquad \Phi_X(z) = e^{i\mu z - \frac{1}{2}\sigma^2 z^2} \tag{1.8}$$

**Example 1.15 (Poisson CF)** *For a Poisson distribution P($\lambda$), we can define the probability mass and characteristic function as:*

$$f(k) := \mathbb{P}(X = k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad \Phi_X(z) = e^{-\lambda(1-e^{iz})} \tag{1.9}$$

## 1.4 Brownian Motion

### 1.4.1 Normal Distribution

The normal distribution, $N(\mu, \sigma^2)$ is (one of) the most important distributions. As seen before, its characteristic function is given by:

$$\Phi_{Normal}(z; \mu, \sigma^2) = e^{i\mu z - \frac{1}{2}\sigma^2 z^2} \tag{1.10}$$

and the density function is:

$$f_{Normal}(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \tag{1.11}$$

The normal, by definition, is symmetric around its mean, has a skewness equal to 0 and a kurtosis equal to 3.

### 1.4.2 Brownian Motion

Brownian motion is the dynamic counterpart - where we work with evolution in time - of the Normal distribution. Brownian motion originates in work of

the botanist Robert Brown in 1828. It was first introduced into finance by Louis Bachelier in 1900, and developed in physics by Albert Einstein in 1905. Brownian motion was first proved mathematically by Norbert Wiener in 1923. In honor of this, Brownian motion is also known as the *Wiener process*.

**Definition 1.16 (Brownian motion)** *A stochastic process $X = (X_t)_{t\geq 0}$ is a standard (one-dimensional) Brownian motion, $W$, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if*

*(i)* $X(0) = 0$, *almost surely,*

*(ii)* $X$ *has independent increments:* $X(t+u)-X(t)$ *is independent of* $\sigma(X(s) : s \leq t)$ *for* $u \geq 0$,

*(iii )* $X$ *has stationary increments: the law of* $X(t+u) - X(t)$ *depends only on* $u$,

*(iv )* $X$ *has Gaussian increments:* $X(t+u) - X(t)$ *is normally distributed with mean 0 and variance* $u$, *i.e.* $X(t+u) - X(t) \sim N(0, u)$,

*(v)* $X$ *has continuous paths:* $X(t)$ *is a continuous function of* $t$, *i.e.* $t \to X(t, \omega)$ *is continuous in* $t$ *for all* $\omega \in \Omega$.

### Filtration for Brownian motion

**Definition 1.17** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a Brownian motion $W_t$, $t \geq 0$. A filtration for the Brownian motion is a collection of $\sigma$-algebras $\mathcal{F}_t$, $t \geq 0$, satisfying:*

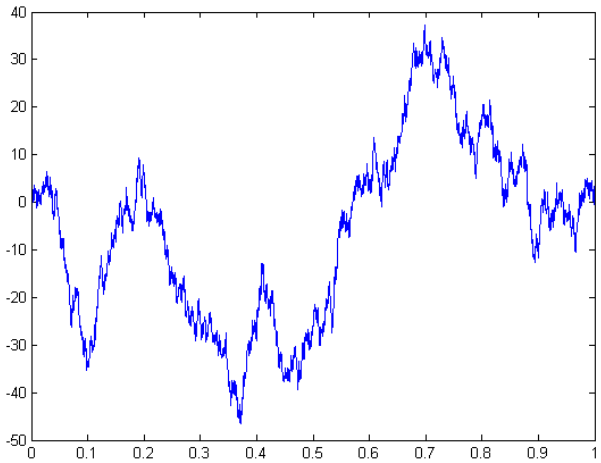



Figure 1.1: Sample path of a standard Brownian motion

*(i)* ***(Information accumulates)*** *For $0 \leq s < t$, every set in $\mathcal{F}_s$ is also in $\mathcal{F}_t$. In other words, there is at least as much information available at the later time $\mathcal{F}_t$ as there is at the earlier time $\mathcal{F}_s$.*

*(ii)* ***(Adaptivity)*** *For each $t \geq 0$, the Brownian motion $W_t$ at time $t$ is $\mathcal{F}_t$ measurable. In other words, the information available at time $t$ is sufficient to evaluate the Brownian motion $W_t$ at that time.*

*(iii)* ***(Independence of future increments)*** *For $0 \leq t < u$, the increment $W_u - W_t$ is independent of $\mathcal{F}_t$. In other words, any increment of the Brownian motion after time $t$ is independent of the information available at time $t$.*

### Properties of Brownian motion

**Definition 1.18 (Martingale property)** *Brownian motion is a martingale.*

$$\begin{aligned}\mathbf{E}[W_t|\mathcal{F}_s] &= \mathbf{E}[(W_t - W_s) + W_s|\mathcal{F}_s] \\ &= \mathbf{E}[W_t - W_s|\mathcal{F}_s] + \mathbf{E}[W_s|\mathcal{F}_s] \\ &= \mathbf{E}[W_t - W_s] + W_s \\ &= W_s \end{aligned} \tag{1.12}$$

**Proposition 1.19 (Path properties)** *Brownian motion has continuous paths, i.e. $W_t$ is a continuous function of $t$. However, the paths of Brownian motion are very erratic; they are nowhere differentiable. The paths of Brownian motion are also of infinite variation, i.e., their variation is infinite on every interval.*

**Definition 1.20 (Brownian scaling)**

If $W_t$ is a Brownian motion, for any $c > 0$,

$$\tilde{W}_t := cW_{t/c^2}, \qquad t \geq 0 \tag{1.13}$$

is also a Brownian motion.

**Theorem 1.21 (Classic Brownian motion martingales)** *Each of the following processes is a continuous martingale with respect to the standard Brownian filtration:*

1. $X_t = W_t$
2. $X_t = W_t^2 - t$
3. $X_t = e^{\alpha W_t - \frac{1}{2}\alpha^2 t}$

## 1.5 Stochastic Integration - Itô Calculus

Stochastic integration was introduced in 1944 by K.Itô in 1944, hence its name Itô calculus. It gives a meaning to

$$\int_0^t X_t dY_t$$

for suitable stochastic processes $X = (X_t, t \geq 0)$ and $Y = (Y_t, t \geq 0)$, the integrand and the integrator. Because we will take as integrator processes of infinite (unbounded) variation on every interval (e.g. Brownian motion, $Y_t = W_t$), stochastic integral can be quite different from classical deterministic integrals.

### 1.5.1 Itô's Lemma

Suppose that $b$ is adapted and locally integrable (so $\int_0^t b(s)ds$ is defined as an ordinary integral), and $\sigma$ is adapted and measurable so that $\int_0^t \sigma(s)dW(s)$ is defined as a stochastic integral. Then

$$X(t) := x_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dW(s) \tag{1.14}$$

defines a stochastic process (or Itô process) $X$ with $X(0) = x_0$. It is customary to express such an equation in differential form, in terms of the stochastic differential equation

$$dX_t = b(t)dt + \sigma(t)dW_t, \qquad X(0) = x_0. \tag{1.15}$$

Now suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, continuously differentiable once in its first argument (which will denote time) and twice in its second (space): $f \in C^{1,2}$. The question arises of giving a meaning to the stochastic differential $df(X_t)$ of the process $f(X_t)$, and finding it.

**Theorem 1.22 (Itô's Lemma)** *If a stochastic process $X_t$ has stochastic differential given by $dX_t = b(t)dt + \sigma(t)dW_t$, then $f = f(t, X_t)$ has stochastic differential*

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dX_t dX_t \tag{1.16}$$

*or reduced to an expression that involves only $dt$ and $dW_t$*

$$df = \left(\frac{\partial f}{\partial t} + b\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}\sigma dW_t \tag{1.17}$$

*since $dW_t dW_t = dt$ and $dtdt = 0$. Or, with $f(0, x_0)$ the initial value of $f$*

$$f = f(0, x_0) + \int_0^t \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}b + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2\right)dt + \int_0^t \frac{\partial f}{\partial x}\sigma dW_t. \tag{1.18}$$

**Proposition 1.23**

$$\mathbf{E}(f(t, X_t) = f(0, x_0) + \int_0^t \mathbf{E}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}b + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\sigma^2\right)dt. \tag{1.19}$$

### 1.5.2 Geometric Brownian Motion

Now we have both Brownian motion $W$ and Itô's Lemma to hand, we can introduce a very important stochastic process, a relative of Brownian motion - *geometric Brownian motion.*

Suppose we wish to model the time evolution of a stock price $S(t)$. Consider how $S$ will change in some small time-interval from the present time $t$ to a time $t + dt$ in the near future. Writing $dS(t)$ for the change $S(t + dt) - S(t)$ in $S$, the return on $S$ in this interval is $dS(t)/S(t)$. Is is economically reasonable to decompose this return into two components, a *systematic* part and a *random* part. The systematic part could be modeled by $\mu dt$, where $\mu$ is some parameter representing the mean rate of return of the stock. The random part could be

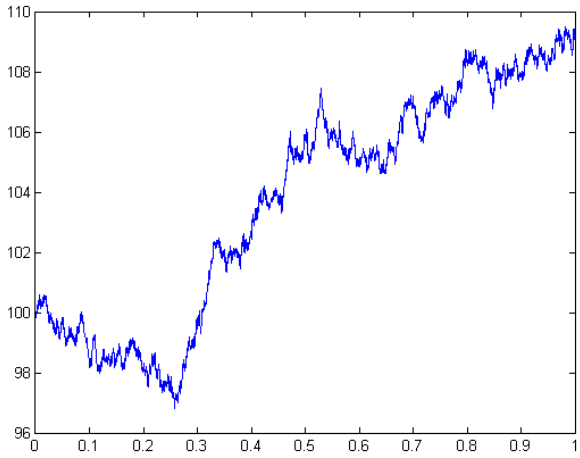



Figure 1.2: Sample path of a geometric Brownian motion

modeled by $\sigma dW(t)$, where $dW(t)$ represents the noise term driving the stock price dynamics, and $\sigma$ is a second parameter describing how much effect this noise has (thus $\sigma$ is called the volatility of the stock).
Putting this together, we have the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t), \qquad S(0) > 0. \tag{1.20}$$

This differential equation has the unique solution

$$S_t = S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t} \tag{1.21}$$

PROOF: Let $(W_t)_{t\geq 0}$ be a Brownian motion, let $(\mathcal{F}_t)_{t\geq 0}$ be an associated filtration, and let $\alpha(t)$ and $\sigma(t)$ be adapted processes. Define the Itô process

$$X_t = \int_0^t \sigma(s)dW_s + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right) ds. \tag{1.22}$$

Then

$$dX_t = \sigma(t)dW_t + \left(\alpha(t) - \frac{1}{2}\sigma^2(t)\right) dt,$$

and

$$dX_t dXt = \sigma^2(t) dW_t dW_t = \sigma^2(t) dt.$$

Consider an asset price process given by

$$S_t = S(0)e^{X_t} = S(0)exp\left\{\int_0^t \sigma(s)dW_s + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right) ds\right\}, \quad (1.23)$$

where $S(0)$ is nonrandom and positive. We may write $S_t = f(X_t)$, where $f(x) = S(0)e^x$, $f'(x) = S(0)e^x$ and $f''(x) = S(0)e^x$. According to the Itô formula

$$\begin{aligned} dS_t &= df(X_t) \\ &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)dX_t dX_t \\ &= S(0)e^{X_t}dX_t + \frac{1}{2}S(0)e^{X_t}dX_t dX_t \\ &= S_t dX_t + \frac{1}{2}S_t dX_t dX_t \\ dS_t &= \alpha(t)S_t dt + \sigma(t)S_t dW_t \end{aligned} \quad (1.24)$$

The asset price $S_t$ has instantaneous mean rate of return $\alpha(t)$ and volatility $\sigma(t)$. Both the mean rate of return and the volatility are allowed to be time-varying and random. *This example includes all possible models of an asset price process that is always positive, has no jumps, and is driven by a simple Brownian motion.* Although the model is driven by a Brownian motion, the distribution of $S_t$ does not need to be log-normal because $\alpha(t)$ and $\sigma(t)$ are allowed to be time-varying and random.

If $\alpha$ and $\sigma$ are constant, we have the usual geometric Brownian motion model $dS_t = S_t(\alpha dt + \sigma dW_t)$, and the distribution of $S_t$ is log-normal.

$$S_t = S(0)exp\left\{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}. \quad (1.25)$$

## 1.6 Mathematical Finance in Continuous Time

### 1.6.1 Trading strategies

Suppose a market with $d$ assets whose prices are described by a stochastic process $S_t = (S_t^1, \ldots, S_t^d)$. We can create a portfolio $\phi = (\phi^1, \ldots, \phi^d)$ composed of a certain amount of each asset. The value of the portfolio is thus given by

$$V_t(\phi) = \sum_{k=1}^{d} \phi^k S_t^k = \phi \cdot S_t. \tag{1.26}$$

A trading strategy consists of maintaining a dynamic portfolio $\phi_t$ by buying and selling assets at different dates $T_0 = 0 < T_1 < T_2 < \ldots < T_n < T$ (between two transaction dates, the portfolio remains unchanged). The portfolio $\phi_t$ held at date $t$ may be expressed as:

$$\phi_t = \phi_0 \mathbf{1}_{t=0} + \sum_{i=0}^{n} \phi_i \mathbf{1}_{]T_i, T_{i+1}]}(t). \tag{1.27}$$

Then, we can define the capital gain of the portfolio up to time $t$ as:

$$\begin{aligned} G_t(\phi) &= \phi_0 S_0 + \sum_{i=0}^{j-1} \phi_i (S_{T_{i+1}} - S_{T_i}) + \phi_j (S_t - S_{T_j}) \quad \text{for } T_j < t \leq T_{j+1} \\ G_t(\phi) &= \int_0^t \phi_u dS_u \end{aligned} \tag{1.28}$$

The last part of the first expression allows to calculate the "current balance" between $T_j$ (the last transaction date) and $t$ (as transaction dates are not continuous in time and $t$ may be fall between two transaction dates).
The difference between the value of the portfolio and its capital gain gives the cost of the strategy up to time $t$:

$$C_t(\phi) = V_t(\phi) - G_t(\phi) = \phi_t S_t - \int_0^t \phi_u dS_u. \tag{1.29}$$

A strategy $\phi$ is said to be self-financing if the cost is equal to zero: the value of the portfolio is then equal to its initial value plus the capital gain accumulated between 0 and $t$:

$$V_t(\phi) = \int_0^t \phi_u dS_u \tag{1.30}$$

### 1.6.2 Risk-Neutral Pricing Pricing and Martingale Measures

#### Pricing Rules

In mathematical finance, two important concepts are the *absence of arbitrage*, which imposes constraints on the way instruments are priced in a market and the notion of *risk-neutral pricing*, which represents the price of any instrument in an arbitrage-free market as its discounted expected payoff under an appropriate probability measure called the "risk-neutral" measure. As these two concepts use the important notion of *equivalent martingale measure*, let's try to understand it better, its construction, its relation to arbitrage pricing and market completeness.

Consider a market whose possible evolutions between 0 and $T$ are described by a scenario space $(\Omega, \mathcal{F})$. $\mathcal{F}$ contains all statements which can be made about behavior of prices between 0 and $T$. $S_t^i(\omega)$ represents the value of asset $i$ at time $t$ in the market scenario $\omega$ and $S_t^0$ is a numéraire.

**Definition 1.24 (Numéraire)** *A numéraire is a price process $S_t^0$ almost surely strictly positive for each $t \in [0, T]$.*

A typical example of numéraire is a cash amount with interest rate $r$: $S_t^0 = e^{rt}$.

Discounting is done using the numéraire $S_t^0$: for any portfolio with value $V_t$, the discounted value is defined by

$$\hat{V}_t = \frac{V_t}{S_t^0}$$

and $B(t,T) = S_t^0/S_T^0$ is called the discount factor. If the numéraire is $S_t^0 = e^{rt}$, then $S_T^0 = e^{rT}$ and the discount factor is $B(t,T) = e^{-r(T-t)}$.

A contingent claim with maturity $T$ can be represented as a payoff at maturity, $H(\omega)$ in each scenario. How can we attribute a value to each contingent claim $H$? A *pricing rule* is a procedure which attributes to each contingent claim a value $\Pi_t(H)$ at each point of time, using the information given at $t$.

For any random payoff, we have thus the following formula:

$$\Pi_0(H) = e^{-rT}\mathbf{E}^{\mathbb{Q}}[H] \tag{1.31}$$

where $\mathbb{Q}$ is called the *risk-neutral measure*. It is important to understand that $\mathbb{Q}$ has nothing to do with the objective probability of occurrence of scenarios. Although it is a probability measure on the set of scenarios, $\mathbb{Q}$ is not the probability that such event happens but the value of a *bet* on the occurrence of this event. A risk-neutral measure is just a convenient representation of the pricing rule $\Pi$: it is not obtained by an econometric analysis of time-series but by looking at prices of contingent claims at $t = 0$.

However, the pricing rule $\Pi$ must be time consistent, i.e. the value at 0 of the payoff $H$ at $T$ is the same as the value at 0 of the payoff $\Pi_t(H)$ at $t$, then $\mathbb{Q}$ should be restrict to $\mathcal{F}_t$ and $\Pi_t(H)$ is given by the discounted conditional expectation with respect to $\mathbb{Q}$:

$$\Pi_t(H) = e^{-r(T-t)}\mathbf{E}^{\mathbb{Q}}[H|\mathcal{F}_t]. \tag{1.32}$$

Now, the problem is to understand which probability measure $\mathbb{Q}$ one shall use, and for this we must examine what restrictions are imposed to $\mathbb{Q}$ by the requirement of absence of arbitrage.

### Arbitrage-Free Pricing

Assume that, in addition to the market scenarios $(\Omega, \mathcal{F})$ and the information flow $\mathcal{F}_t$, we know something about the probability of occurrence of these scenarios, represented by a probability measure $\mathbb{P}$, which represents the objective probability of future scenarios.
A fundamental requirement for a pricing rule is the absence of arbitrage opportunities. As seen before, an arbitrage opportunity is a self-financing strategy $\phi$ which can lead to a positive terminal gain, but with probability 0 to obtain a loss.

**Definition 1.25 (Equivalent Probability Measures)** *The probabilities* $\mathbb{Q}$ *and* $\mathbb{P}$ *are said to be equivalent if they define the same set of (im)possible events (null set):*

$$\mathbb{P} \sim \mathbb{Q} : \forall A \in \mathcal{F} \qquad \mathbb{Q}(A) = 0 \Longleftrightarrow \mathbb{P}(A) = 0. \tag{1.33}$$

Let $S^i$ be an asset traded at price $S^i_t$. This asset can be held until $T$, generating a terminal payoff $S^i_T$, or be sold for $S^i_t$: the resulting sum invested at the interest rate $r$ will then generate a terminal wealth of $e^{r(T-t)}S^i_t$. these two buy-and-hold strategies are self financing and have the same terminal payoff so, according to the *law of one price*, they should have the same value at $t$:

$$\mathbf{E}^{\mathbb{Q}}[S^i_T|\mathcal{F}_t] = \mathbf{E}^{\mathbb{Q}}[e^{r(T-t)}S^i_t|\mathcal{F}_t] = e^{r(T-t)}S^i_t. \tag{1.34}$$

Dividing by $S_T^0 = e^{rT}$ we obtain:

$$\mathbf{E}^{\mathbb{Q}}\left[\frac{S_T^i}{S_T^0}|\mathcal{F}_t\right] = \frac{S_t^i}{S_t^0}, \quad i.e., \quad \mathbf{E}^{\mathbb{Q}}[\hat{S}_T^i|\mathcal{F}_t] = \hat{S}_t^i. \tag{1.35}$$

Therefore absence of arbitrage implies that discounted values $\hat{S}_t^i = e^{-rt}S_t^i$ of all traded assets are martingales with respect to the probability measure $\mathbb{Q}$. A probability measure verifying these two equations is called an *equivalent martingale measure*. Any arbitrage-free pricing rule is thus given by an equivalent martingale measure. Conversely, it could be easily shown that any equivalent martingale measure $\mathbb{Q}$ defines an arbitrage-free pricing rule. There is hence a one-to-one correspondence between arbitrage-free pricing rules and equivalent martingale measure.

**Definition 1.26 (Equivalent Martingale Measure)** *We say that a probability measure $\mathbb{Q}$ on the market scenarios is an equivalent martingale measure if:*

*(i) $\mathbb{Q}$ is equivalent to $\mathbb{P}$, i.e. they have the same null sets (events which cannot happen under $\mathbb{P}$ also cannot happen under $\mathbb{Q}$ and vice versa),*

$$\mathbb{P} \sim \mathbb{Q}: \qquad \forall A \in \mathcal{F} \quad \mathbb{Q}(A) = 0 \Longleftrightarrow \mathbb{P}(A) = 0 \tag{1.36}$$

*(ii) the discounted stock-price process $\hat{S}_t = e^{-rT}S_t$, $t \geq 0$ is a martingale under $\mathbb{Q}$.*

$$\mathbf{E}^{\mathbb{Q}}[\hat{S}_T^i|\mathcal{F}_t] = \hat{S}_t^i \tag{1.37}$$

**Definition 1.27 (Risk-Neutral Pricing)** *In a market described by a probability measure $\mathbb{P}$ on scenarios, any arbitrage-free pricing rule $\Pi$ can be represented as*

$$\Pi_t(H) = e^{-r(T-t)}\mathbf{E}^{\mathbb{Q}}[H|\mathcal{F}_t], \tag{1.38}$$

*where $\mathbb{Q}$ is an equivalent martingale measure.*

**Theorem 1.28 (Fundamental Theorem of Asset Pricing)** *The market model defined by $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ and asset prices $(S_t)_{t\in[0,T]}$ is arbitrage-free if and only if there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that the discounted assets $(\hat{S}_t)_{t\in[0,T]}$ are martingales with respect to $\mathbb{Q}$.*

### 1.6.3 Market Completeness

Besides the idea of absence of arbitrage, another important concept originating the Black-Scholes model is the concept of *perfect hedge*: a self-financing strategy $(\phi_t^0, \phi_t)$ is said to be a perfect hedge (or a replication strategy) for a contingent claim $H$ if:

$$H = V_0 + \int_0^T \phi_t dS_t + \int_0^T \phi_t^0 dS_t^0 \tag{1.39}$$

A market is said *complete* if any contingent claim admits a replicating portfolio. In a complete market, there is only one way to define the value of a contingent claim: the value of any contingent claim is given by the initial capital needed to set up a perfect hedge for $H$. All equivalent martingale measures give the same pricing rules, therefore we have the following theorem:

**Theorem 1.29 (Fundamental Theorem of Asset Pricing, $2^{nd}$)** *A market defined by the assets $(S_t^0, S_t^1, \ldots, S^d)_{t\in[0,T]}$, described as stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ is complete if and only if there exists a unique martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$.*

# Chapter 2

# Black-Scholes Model and its Limitations

## 2.1 The Black-Scholes Model

The theory of mathematical finance began in 1900 when the french mathematician Louis Bachelier, in his thesis *Théorie de la spéculation* [7], proposed the following model to describe the price $S$ of an asset at the Paris Bourse:

$$S_t = S_0 + \sigma W_t$$

where $W_t$ is a Brownian motion.
However, this model had many imperfections, including, for example, negative stock prices. A more appropriate model was thus suggested by Samuelson in 1965 [85]: geometric Brownian motion in which log-price follows a Brownian motion.
In 1973, Black, Scholes [19] and Merton [73], in their famous papers, demon-

strated how to price a European call option based on this model. Indeed, they assume the stock price follows a geometric Brownian motion and give some conditions to derive the option pricing formula:

1. There are no transaction costs or taxes, trading takes place continuously in time and borrowing and short-selling are allowed (the market is frictionless).

2. The short-term interest rate (the risk-free rate $r$) is known and constant through time.

3. The stock pays no dividend during the life of the option.

4. The option is European (it can only be exercised at the expiration date).

5. The stock price follows a geometric Brownian motion through time which produces a log-normal distribution for stock price between any two points in time.

6. The volatility is constant for any strike and maturity.

It has been shown that the model can be easily modified when the interest rate is stochastic or a function of $t$, when the stock gives dividend or when the option is American.
Because of its simplicity and its independence of investors expectations about future asset returns, the Black-Scholes formula is widely used among practitioners for pricing and hedging options.

In the Black-Scholes world, the stock price, $S$, follows a geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{2.1}$$

where $\mu$ and $\sigma$ are known constants, $W_t$ is a standard Brownian motion. It can be shown that the solution of this stochastic differential equation is

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t} \tag{2.2}$$

The essential step in the Black-Scholes methodology is the construction of a risk-less portfolio. Then, based on the no-arbitrage argument, a partial differential equation can be derived for the price of a call option. This partial differential equation can be easily solved and gives a closed-form solution. The main derivation goes as follows.

Suppose that $C$ is the price of a call option or other derivative contingent on $S$. By Ito's lemma,

$$dC = \left(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial C}{\partial S}\sigma S dW_t. \tag{2.3}$$

Next, we set up a portfolio consisting of a short position in a call option and a long position of $\Delta$ units of stock. Define $\Pi$ as the value of the portfolio,

$$\Pi = -C + \Delta S. \tag{2.4}$$

The change in the value of this portfolio in a small time interval is given by

$$d\Pi = -dC + \Delta dS. \tag{2.5}$$

Substituting (2.1), (2.3) into (2.5) yields

$$\begin{aligned} d\Pi &= -\left(\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\right) dt - \frac{\partial C}{\partial S}\sigma S dW_t + \Delta\mu S dt + \Delta\sigma S dW_t \\ &= \left(-\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2} + \Delta\mu S\right) dt + \left(-\frac{\partial C}{\partial S}\sigma S + \Delta\sigma S\right) dW_t \end{aligned}$$

To make the portfolio risk-less, we choose $\Delta = \frac{\partial C}{\partial S}$. Then

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right) dt. \tag{2.6}$$

We can see that making the portfolio risk-less, we have removed any source of randomness (the Brownian motion $W_t$) and the value of the portfolio is now deterministic.

On the other hand, in the absence of arbitrage opportunities, this risk less portfolio must earn a risk-free rate, $r$,

$$d\Pi = r\Pi dt. \tag{2.7}$$

Substituting from (2.4) and (2.6), this becomes

$$\left(-\frac{\partial C}{\partial t} - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right) dt = r\left(-C + \frac{\partial C}{\partial S}S\right) dt,$$

or

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} = rC. \tag{2.8}$$

Equation (2.8) is the famous Black-Scholes partial differential equation. The solution depends on the boundary conditions. In the case of a European vanilla call option, the final condition is that the option price is simply its payoff at maturity.

$$C = max(S - K, 0), \qquad t = T, \tag{2.9}$$

where $K$ is the strike price. So the equation can be solved by backward in time with the final condition. The Black-Scholes formula for the European call option is

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2), \tag{2.10}$$

where

$$d_1 = \frac{ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2 = \frac{ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}, \tag{2.11}$$

and $S_0$ is the current stock price, $T$ is the time to maturity, $\sigma$ is the stock price volatility, $N(x)$ is the cumulative probability distribution function for the standard normal distribution. The price of the European put option can be computed by the *Put-Call parity*.

$$C + Ke^{-rT} = P + S_0. \tag{2.12}$$

The expected return $\mu$ does not appear in the Black-Scholes equation. This means the pricing formula is independent of the individual's preference. This amazing property together with its simplicity makes the Black-Scholes pricing formula very popular among practitioners.

Another approach to derive the Black-Scholes formula is the *risk-neutral valuation* method. The price of the option, $C$, is the expected value of the option at maturity in a risk-neutral world discounted at the risk-free rate, that is

$$C = e^{-rT} \mathbf{E}^{\mathbb{Q}}[max(S_T - K, 0)], \tag{2.13}$$

where $\mathbf{E}^{\mathbb{Q}}$ denotes the expected value in a risk-neutral world. $\mathbb{Q}$ is also called the *equivalent martingale measure*. In the risk-neutral world,

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_t} \tag{2.14}$$

So, the expectation in (2.13) can be calculated by integrating over the normal distribution. We can get the same pricing formulas as (2.10) and (2.11). The above two pricing methods, *no-arbitrage valuation* and *risk-neutral valuation*, are two general approaches to pricing options in modern finance literature.

## 2.2 The Limitations of the Black-Scholes Model

### 2.2.1 The Evidence of the Volatility Smile

Although the Black-Scholes formula is powerful to price stock options and simple to use, many empirical results show that it may systematically misprice a number of options. The most well-known phenomenon related to the biases of the Black-Scholes model is the so-called volatility smile or skew. The implied volatilities from the market prices of options tend to vary by strike prices and maturities.

The implied volatility is the volatility used in the Black-Scholes model such that the observed market price of the option equals the model price.

$$c^{BS}(\sigma) = c^{Market} \tag{2.15}$$

Consider call or put options on a given stock or an index. We take options with the same maturity but different strike prices. We apply the Black-Scholes

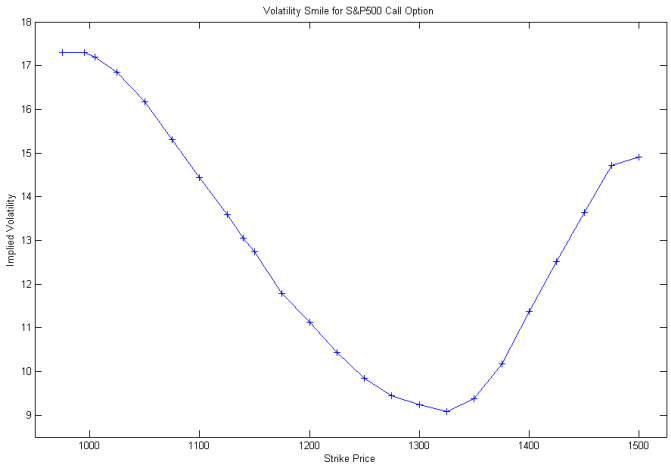



Figure 2.1: Volatility smile. Implied volatilities for S&P 500 call options. Maturity is Jun. 17, 2005. Valuation date is Feb. 24, 2005. the S&P 500 index is 1200.20 on the valuation day. Use $r = 0.011$.

model to back out the implied volatilities and plot them against the strike prices. We expect the implied volatilities to be identical because the constant volatility is one of the assumptions of the Black-Scholes model. However, it is likely not the case in practice. Most option markets exhibit persistent patterns of non-constant volatilities. In some markets, the implied volatilities form a "U-shape", which is called the volatility smile. In-the-money options and out-of-the-money options have higher implied volatilities than at-the-money options. Generally, the shape of the volatility smile is not symmetric. It is more a skewed curve. People also call it volatility skew or volatility smirk. Usually, the smile is significant for short maturity options and tends to be flat for long maturity options. The volatility smile is caracteristic of derivatives on currencies and the volatility skew of derivatives on stocks or index.

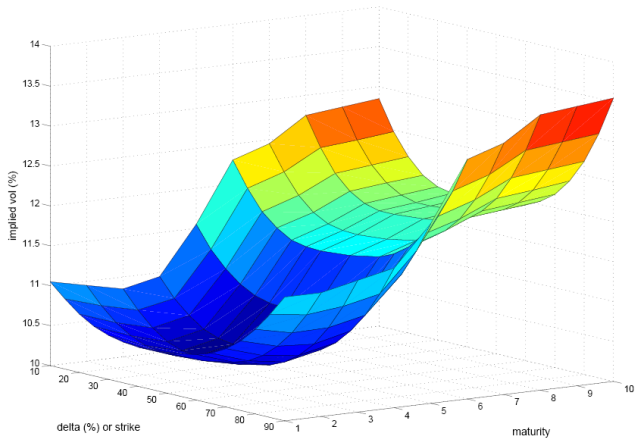



Figure 2.2: Volatility surface. Implied volatilities of vanilla options on the EUR/USD exchange rate on Nov. 5, 2001.

In addition to calculating a volatility smile, we can also calculate a volatility term structure, a function of maturity for a fixed strike price. The implied volatilities also vary with maturity. Combining the volatility smile and the volatility term structure, we can generate a volatility surface, one dimension for strike price and the other for maturity.

There are various explanations for the phenomenon of volatility smile. Some of explanations are related to the idealized assumptions of the Black-Scholes model which says the asset price follows a geometric Brownian motion with a constant volatility. Other explanation of this smile is that it is caused by strong demand for slightly out-of-the-money put options. A fund manager has his performance reviewed every three months. He wants to be protected against the possibility of a market crash in the mean time. Therefore, he buy put options which guarantee that his portfolio's value can only fall by a small amount even if the market crashes. Thus he is buying the put option as insur-

ance. Since there are many fund managers doing the same thing, hedging is impossible, and no one wants extra exposure to crashes, the market is all one way, and the price of the put option is bid up.
Obviously, the phenomenon of volatility smile is not consistent with the Black-Scholes model.

### 2.2.2 The Incompleteness of Markets

The Black-Scholes model assumes that the market is complete, i.e. that any contingent claim admits a replicating portfolio, hence it can be perfectly hedged. However, while most stochastic models used in option pricing are arbitrage-free, only a few of these are complete: stochastic volatility and exponential-Lévy models (as we will see shortly) are examples of incomplete models.
But, is market completeness so important for modeling option prices? Everybody knows that perfect hedge cannot exist in practice: all risks cannot be hedged. Some people talked about the inconsistency of the idea of "frictionless" market with the reality (no costs, continuous trading available, etc.) but this feature represents only a very little part of the risk that someone doesn't take in consideration with a diffusion model. *Dynamic hedging* represents the basis of the fundamental idea of Black-Scholes model: *Delta hedging*. Dynamic hedging allows to continuously modify your position in option to let unchanged the Delta ratio. Hence, the *unique* risk admitted by Black-Scholes model, *delta risk* can be easily hedged.
However, taking positions in options induces other market risks, as *gamma risk* or *vega risk* (the "volatility" risks), that a diffusion model doesn't even

mention. The motivation for using jumps in a model is that stock markets do crash and during a crash there is no opportunity to carry out a continuously-changing Delta hedge. One consequence of this will be the impossibility of perfect hedging: at a given time the stock price can increase slightly or decrease slightly or fall a lot. It is not possible to be hedged against all of these simultaneously. The impossibility of perfect hedging means that the market is incomplete, that is not every option can be replicated by a self-financing portfolio. Hence, it makes much more sense to use incomplete market models where the risk of hedging can be quantified rather than sticking to complete market models where the risk of hedging is by definition zero.

### 2.2.3 Are these Returns Really Log-normally Distributed?

If the stock price follows a geometric Brownian motion as in equation (2.1), the stock price is log-normally distributed, or, the logarithmic return is normally distributed. From figure 2.3, we see that price changes are small in sequential days in some periods and large in other periods. This is called volatility clustering. This feature implied that the volatility is autocorrelated.
Then, if confronted with figure 2.4, it is easy to see that the returns of the NASDAQ are clearly not log-normally distributed.
Empirical studies find stock returns have a higher kurtosis (higher central peak and fatter tails) compared to the normal distribution which is assumed by the Black-Scholes model. Figure 2.5 shows clearly that pattern (the distribution of the NASDAQ returns has an excess kurtosis of 2.5817). Leptokurtosis is consistent with the volatility smile.

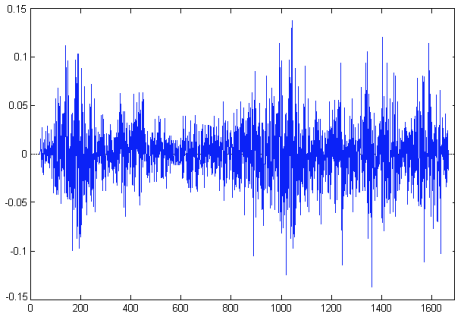



Figure 2.3: NASDAQ daily returns from Nov. 16, 1999 to Nov. 16, 2006

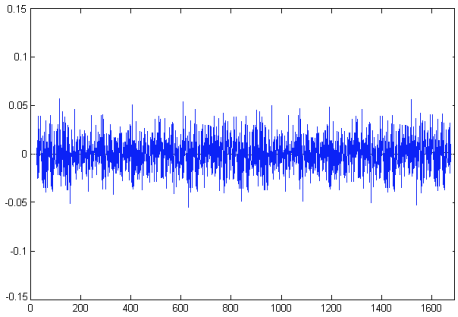



Figure 2.4: Returns of a simulated geometric Brownian motion

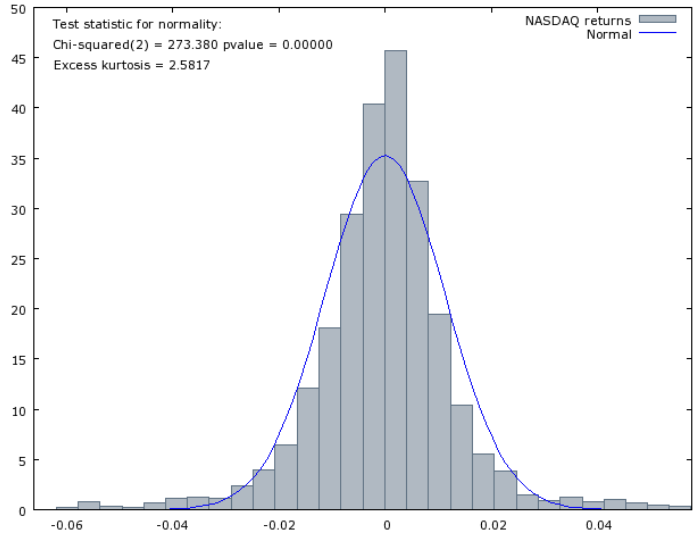



Figure 2.5: NASDAQ returns and normal distributions

### 2.2.4 New Models, New Solutions, New Problems

Recent developments in mathematical finance have explored several way to model the underlying. One of the mains goals of these models is to capture the *Volatility Smile*.

#### The Volatility Matrix

The simplest way to incorporate the volatility smile is to use the volatility matrix. Market prices of options are used to generate implied volatilities. The volatility matrix replicates the volatility surface. When we want to price a new option, we can pick up a corresponding volatility and apply the Black-Scholes model to get the price.

One way to modify the constant volatility assumption of the Black-Scholes model is to assume that volatility is a deterministic function of time and stock

price: $\sigma = \sigma(t, S)$. In the special case of $\sigma = \sigma(t)$, a deterministic function of time, the option price will just be the Black-Scholes formula with volatility $\sqrt{\overline{\sigma}^2}$, where $\overline{\sigma}^2 = \frac{1}{T}\int_0^T \sigma^2(s)ds$.

### Constant Elasticity of Variance

Another famous model with deterministic volatility is the *constant elasticity of variance* (CEV) model proposed by Cox and Ross in 1976, [35]. The stock price in this model is

$$dS = \mu S dt + \sigma S^{\alpha} dW_t, \tag{2.16}$$

where $\alpha$ is a positive constant. So the stock price has volatility $\sigma S^{\alpha-1}$. When $\alpha = 1$, we have the Black-Scholes case. When $\alpha < 1$, the volatility increases as the stock price decreases. This can generate a distribution with a fatter left tail and a thinner right tail. When $\alpha > 1$, the situation is reversed. So the volatility smile can be incorporated in this model. Cox and Ross also provide valuation formulas for European call and put options in the CEV model. Several studies have reported that the CEV model outperforms the Black-Scholes model in most cases. The problem is that the option price in the CEV model will approach either zero or infinite in the long run.

### Local Volatility Models

Derman and Kani, in 1994, [38], [39], proposed a so-called *implied tree* model, which also assumes that the volatility is a deterministic function of stock price

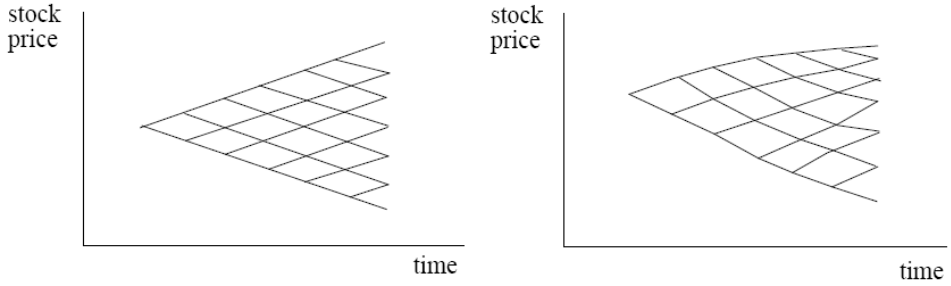



Figure 2.6: A risk-neutral stock tree with constant volatility and an implied tree.

and time.

$$\frac{dS_t}{S_t} = \mu dt + \sigma(t, S_t)dW_t \tag{2.17}$$

The implied tree model is also called the *local volatility* model. More generally, this type of model incorporate the volatility smile in the construction of the tree (see figure 2.6). And this tree is particularly used for pricing exotic options as their prices are consistent with all traded vanilla options. Deterministic volatility models allow volatility to change in a deterministic way.

### Stochastic Volatility Models

But empirical evidence shows the variance of the stock returns is no stationary. The relation between the volatility and the stock or time changes with time. A trader can model the volatility as a function of the stock price and time this week, but next week, this function will be quite different. Hence, it is not enough to allow volatility to change deterministically. Subsequent researches model the volatility as a stochastic variable. In stochastic volatility models,

volatility is modeled as a separate stochastic process. A general representation of the continuous-time stochastic volatility model may be written as

$$dS_t = \mu S_t dt + \sigma_t S_t W_{1t}, \tag{2.18}$$

$$\sigma_t = f(Y_t), \tag{2.19}$$

$$dY_t = a(t, Y_t)dt + b(t, Y_t)dZ_t, \tag{2.20}$$

$$dW_{1t}dZ_t = \rho dt. \tag{2.21}$$

Here, the drift $\mu$ is still a constant. $\sigma_t$ is the volatility of the stock price. $f$ is some positive function. $Y_t$ is some underlying process which determines the volatility. $W_{1t}$ and $Z_t$ are two correlated standard Brownian motions. The constant parameter $\rho$ is the correlation coefficient between these two Brownian motions. We can also rewrite $Z_t$ as

$$Z_t = \rho W_{1t} + \sqrt{1-\rho^2}W_{2t}, \tag{2.22}$$

where $W_{2t}$ is a standard Brownian motion independent of $W_{1t}$. There are some economic arguments for a negative correlation between stock price and volatility shocks.

This general stochastic volatility model contains many famous models, we give three examples:

1. **Hull-White -** Hull and White (1987), [61] assume a geometric Brownian motion for the variance,

   $$dY_t = \alpha Y_t dt + \beta Y_t dZ_t \tag{2.23}$$

   where $\alpha$ and $\beta$ are constants, $f(y) = \sqrt{y}$. This is the first stochastic volatility model for pricing options in financial literature.

2. **Stein-Stein -** Stein and Stein (1991), [92] assume the driving process $Y_t$ is an Ornstein-Uhlenbeck (OU) process,

$$dY_t = \alpha(\omega - Y_t)dt + \beta dZ_t. \tag{2.24}$$

   It is a mean-reverting process. From econometric studies, people believe that volatility is mean-reverting. However, it is not appropriate to simply assume that the volatility is an OU process, because $Y_t$ can be negative in OU process, so they assume $f(y) = |y|$.

3. **Heston -** Heston (1993), [58] assumes $Y_t$ follows a Cox-Ingersoll-Ross (CIR) process,

$$dY_t = \kappa(\theta - Y_t)dt + \xi\sqrt{Y_t}dZ_t, \tag{2.25}$$

   and $f(y) = \sqrt{y}$. $\theta$ is the long-run variance, $\kappa$ is the rate of mean reversion, $\xi$ is called *volatility of volatility.* $Y_t$ is strictly positive when $2\kappa\theta \geq \xi^2$ and non-negative when $0 \leq 2\kappa\theta < \xi^2$. This model is very important because it provides a closed-form formula for the European option and $\rho$ can be non-zero.

### Jump Processes

Finally, another important element in option pricing models is modeling stock prices with jumps. Jump (Lévy) processes have become increasingly popular in mathematical finance because they can describe the observed reality of financial markets in a more accurate way than basic diffusion models based on Brownian motion. In the "real" world, we observe that asset price processes have jumps or spikes and risk-managers have to take them into account. The

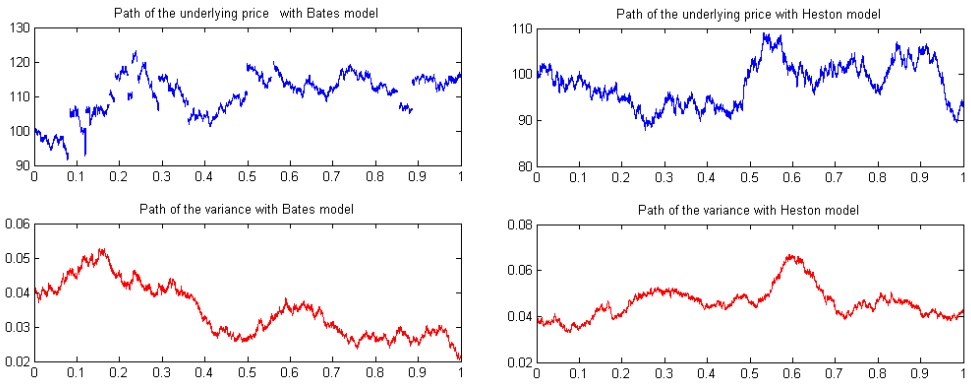



Figure 2.7: Sample paths of the stock price and variance under the Bates and Heston models

figure 2.8, which represents Standard and Poors Index prices on March 07-28, 2008, is a classic example of jumps in stock prices. During the speculative bubble, as shown in figure 2.9, Yahoo! stock was characterized by heavy spikes. NASDAQ stocks, as Microsoft (see figure 2.10) or Cisco (figure 2.11), are characterized by frequent jumps in prices.

Merton, in 1976, [74], added random jumps to the geometric Brownian motion. The stochastic process for the stock price is

$$\frac{dS}{S} = (\mu - \lambda k)dt + \sigma dW + dp, \tag{2.26}$$

where $\lambda$ is the average number of jumps in one period, $k$ is the average jump size, $dp$ is the Poisson process generating the jumps. This jump-diffusion model is useful when the underlying asset price has large changes, because continuous-time models cannot capture this property.

Some other researchers even model the stock price as a pure jump process.


Figure 2.8: Standard and Poors Index prices (March 07-28, 2008, 10-min bars)

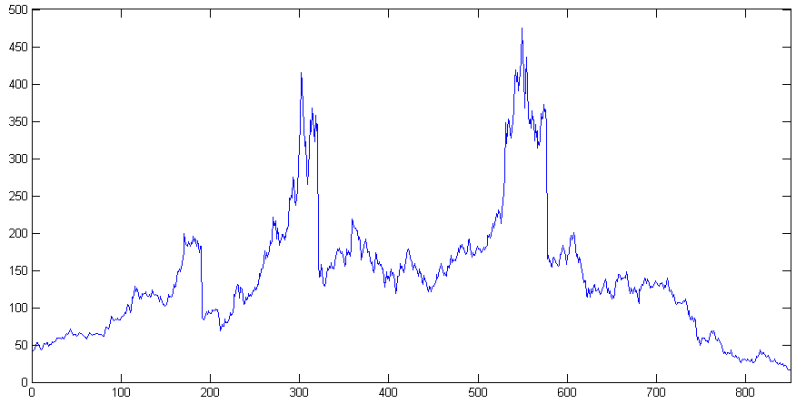



Figure 2.9: Yahoo! stock price during the speculative bubble (1999-2001)

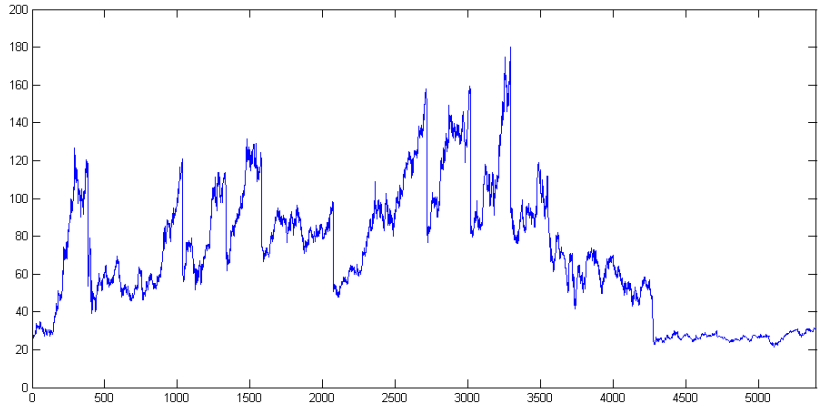



Figure 2.10: Microsoft stock price since the IPO (1986-2007)

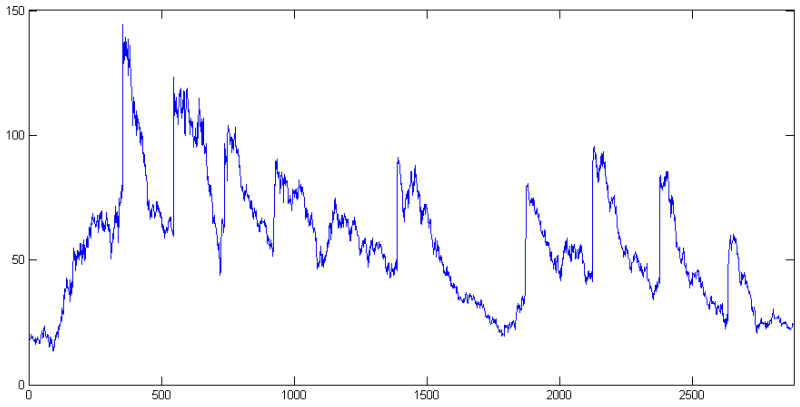



Figure 2.11: Cisco stock price from 1990 (IPO) to 2001

They also combine stochastic volatility and jump-diffusion (Bates, 1996, [13]). We will focus on jump processes in the next chapters.

# Chapter 3

# Lévy Processes

As seen before, asset price time-series present jumps and spikes and the empirical distribution of asset returns exhibits fat tails and skewness, behavior that deviates from normality. Moreover, the implied volatilities are constant neither across strike nor across maturities. Hence, models that accurately fit return distributions are essential. Lévy processes provide us with the appropriate framework to adequately describe all these features.

Processes with independent and stationary increments are name *Lévy processes* after the French mathematician Paul Lévy (1886-1971), who made the connection with infinitely divisible laws, characterized their distributions (Lévy-Khintchine formula) and described their structure (Lévy-Itô decomposition).

## 3.1 Poisson Processes

The fundamental pure jump process is the *Poisson process*. All jumps of a Poisson process are of size one. A *compound Poisson process* is like a Poisson process, except that the jumps are of random size.

### 3.1.1 Poisson Process

In the way that Brownian motion is the basic building block for continuous-path processes, the Poisson process serves as the starting point for jump processes.

#### Exponential Random Variables

We say that a positive variable $\tau$ follows an *exponential distribution* with parameter $\lambda > 0$ if it has a probability density function

$$\lambda e^{-\lambda t} \mathbf{1}_{t \geq 0} \tag{3.1}$$

and the expected value of $\tau$ is $\mathbf{E}(\tau) = \frac{1}{\lambda}$.

The distribution function is given by

$$\forall t \in [0, \infty] \qquad F(t) = \mathbb{P}(\tau \leq t) = 1 - e^{-\lambda t} \tag{3.2}$$

The exponential distribution has an important property called memorylessness:

$$\forall t, s > 0, \qquad \mathbb{P}(T > t + s | T > t) = \mathbb{P}(T > s). \tag{3.3}$$

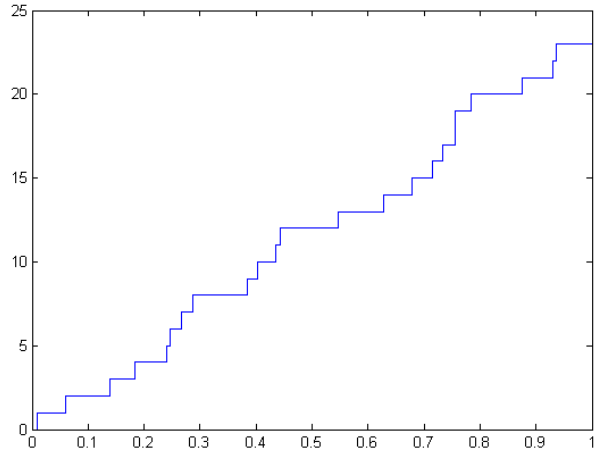



Figure 3.1: Sample path of a Poisson process. $\lambda = 25$.

### Poisson Distribution

An integer valued random variable $N$ is said to follow a Poisson distribution with parameter $\lambda$ if

$$\forall n \in \mathbb{N}, \qquad \mathbb{P}(N = n) = e^{-\lambda}\frac{\lambda^n}{n!} \tag{3.4}$$

### Poisson Process

To construct a Poisson process, we begin with a sequence $\tau_1, \tau_2, \ldots$ of independent exponential random variables, all with the same mean $\frac{1}{\lambda}$. We will build a model in which an event, which we call a "jump", occurs from time to time. The first jump occurs at time $\tau_1$, the second occurs $\tau_2$ time units after the first, the third occurs $\tau_3$ time units after the second, etc. The $\tau_k$ random variables are called the *interarrival times*. The *arrival times* are

$$S_n = \sum_{k=1}^{n} \tau_k \tag{3.5}$$

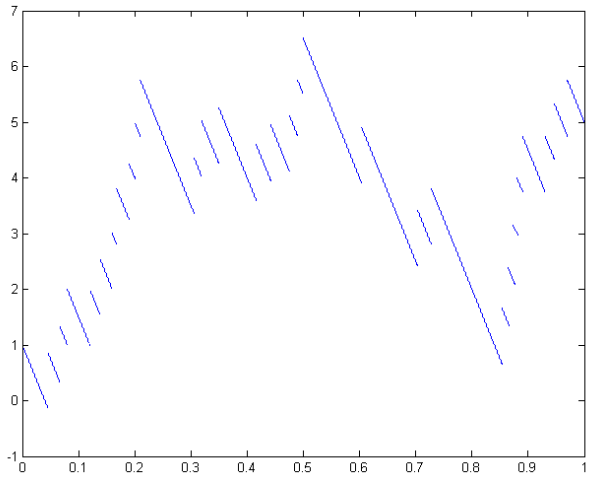



Figure 3.2: Sample path of a compensated Poisson process. $\lambda = 25$.

(i.e., $S_n$ is the time of the $n$th jump). The *Poisson process* $N_t$ counts the number of jumps that occur at or before time $t$

$$N_t = \sum_{n \geq 1} \mathbf{1}_{t \geq T_n} \tag{3.6}$$

The Poisson process is therefore defined as a *counting* process.
The sample paths $t \mapsto N_t$ are càdlàg. $N_t$ has independent increments, and these increments are homogeneous. $N_t$ has the Markov property $\mathbf{E}[f(N_t)|N_u, u \leq s] = \mathbf{E}[f(N_t)|N_s]$, $\forall t > s$. However, the Poisson process is not a martingale. The characteristic function of $N_t$ is given by

$$\Phi_{N_t}(u) = \mathbf{E}[e^{iuN_t}] = e^{\lambda t(e^{iu}-1)}, \qquad \forall u \in \mathbb{R} \tag{3.7}$$

### 3.1.2 Compensated Poisson Process

Let $N_t$ be a Poisson process with intensity $\lambda$. We define the *compensated Poisson process*

$$\tilde{N}_t = N_t - \lambda t. \tag{3.8}$$

Then, $\tilde{N}_t$ is a martingale.

PROOF: Let $0 \leq s \leq t$ be given. Because $N_t - N_s$ is independent of $\mathcal{F}_s$ and has expected value $\lambda(t-s)$, we have:

$$\begin{aligned} \mathbf{E}[\tilde{N}_t|\mathcal{F}_s] &= \mathbf{E}[\tilde{N}_t - \tilde{N}_s|\mathcal{F}_s] + \mathbf{E}[\tilde{N}_s|\mathcal{F}_s] \\ &= \mathbf{E}[N_t - N_s - \lambda(t-s)|\mathcal{F}_s] + \tilde{N}_s \\ &= \mathbf{E}[N_t - N_s] - \lambda(t-s) + \tilde{N}_s \\ &= \tilde{N}_s. \end{aligned} \tag{3.9}$$

The characteristic function is given by

$$\Phi_{\tilde{N}_t}(u) = e^{\lambda t(e^{iu}-1-iu)} \qquad \forall u \in \mathbb{R} \tag{3.10}$$

### 3.1.3 Compound Poisson Process

**Definition 3.1 (Compound Poisson Process)** *A compound Poisson process with intensity $\lambda > 0$ and jump size distribution $f$ is a stochastic process $X_t$ defined as*

$$X_t = \sum_{i=1}^{N_t} Y_i \tag{3.11}$$

*where jumps sizes $Y_i$ are i.i.d. with distribution $f$ and $(N_t)$ is a Poisson process with intensity $\lambda$, independent from $(Y_i)_{i\geq 1}$.*

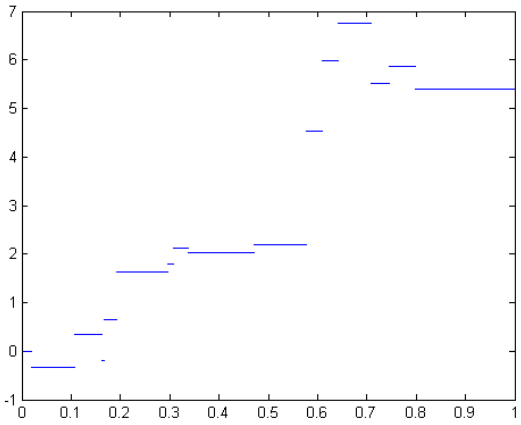



Figure 3.3: Sample path of a compound Poisson process. $\lambda = 25$.

### Properties of the Compound Poisson Process

The following properties of a compound Poisson process are easily deduced from the definition:

(i) The sample paths of $X$ are càdlàg piecewise constant functions.

(ii) The jump times $(T_i)_{i\geq 1}$ have the same law as the jump times of the Poisson process $N_t$: they can be expressed as partial sums of independent exponential random variable with parameter $\lambda$.

(iii) The jump size $(Y_i)_{i\geq 1}$ are independent and identically distributed with law $f$.

**Proposition 3.2** *The Poisson process itself can be seen as a compound Poisson process on* $\mathbb{R}$ *such that* $Y_i := 1$.

The characteristic function of a compound Poisson process $(X_t)_{t\geq 0}$ on $\mathbb{R}$ has the following representation:

$$\begin{aligned}
\Phi_{X_t}(u) &= \mathbf{E}[e^{iu\sum_{k=1}^{N} J_k}] \\
&= \sum_{n\geq 0} \mathbf{E}[e^{iu\sum_{k=1}^{N} J_k}|N=n]P(N=n) \\
&= \sum_{n\geq 0} \mathbf{E}[e^{iu\sum_{k=1}^{N} J_k}]e^{-\lambda}\frac{\lambda^n}{n!} \\
&= \sum_{n\geq 0} \left(\int_{-\infty}^{+\infty} e^{iux}F(dx)\right)^n e^{-\lambda}\frac{\lambda^n}{n!} \\
&= \exp\left(t\lambda\int_{-\infty}^{+\infty}(e^{iux}-1)F(dx)\right) \qquad (3.12)
\end{aligned}$$

where $\lambda$ denotes the jump intensity and $F$ the jump size distribution.
Comparing the characteristic function of the compound Poisson process with those of the Poisson process, we see that a compound Poisson random variable can be represented as a superposition of independent Poisson processes with different jump sizes. The total intensity of Poisson processes with jump sizes in the interval $[x, x+dx]$ is determined by the density $\lambda f(dx)$.

Introducing a new measure $\nu(A) = \lambda f(A)$, we can rewrite the characteristic function of the compound Poisson process as follows:

$$\Phi_{X_t}(u) = \mathbf{E}[e^{iuX_t}] = \exp\left\{t\int_{-\infty}^{+\infty}(e^{iux}-1)\nu(dx)\right\} \qquad \forall u\in\mathbb{R}. \qquad (3.13)$$

As we will see in the following section, $\nu$ is called the *Lévy measure* of the process $(X_t)_{t\geq 0}$. $\nu$ is a positive measure on $\mathbb{R}$ but not a probability measure since $\int \nu(dx) = \lambda \neq 1$.

**Theorem 3.3 (Compensated Compound Poisson Process)** *Let $X_t$ be the compound Poisson process defined above and $\beta$ the average jump size. Then,*

*the compensated compound Poisson process*

$$\tilde{X}_t = X_t - \beta\lambda t \tag{3.14}$$

*is a martingale.*

PROOF: Let $0 \leq s \leq t$ be given. Because the increment $X_t - X_s$ is independent of $\mathcal{F}_s$ and has mean $\beta\lambda(t-s)$, we have

$$\begin{aligned} \mathbf{E}[X_t - \beta\lambda t|\mathcal{F}_s] &= \mathbf{E}[X_t - X_s|\mathcal{F}_s] + \mathbf{E}[X_s|\mathcal{F}_s] - \beta\lambda t \\ &= \beta\lambda(t-s) + X_s - \beta\lambda t \\ &= X_s - \beta\lambda s. \end{aligned} \tag{3.15}$$

## 3.2 Lévy Processes: Definition and properties

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ be a filtered probability space which satisfies the usual conditions.

**Definition 3.4 (Lévy Process)** *A càdlàg, adapted, real valued stochastic process $L = (L_t)_{t\geq 0}$ with $L_0 = 0$ almost surely, is called a* Lévy process *if the following conditions are satisfied:*

*(i)* *$L$ has* independent increments*, i.e. $L_t - L_s$ is independent of $\mathcal{F}_s$ for any $0 \leq s < t \leq T$.*

*(ii)* *$L$ has* stationary increments*, i.e. for any $s, t \geq 0$ the distribution of $L_{t+s} - L_t$ does not depend on $t$.*

*(iii)* *$L$ is* stochastically continuous*, i.e. for every $\epsilon > 0$: $lim_{h\to 0}\mathbb{P}(|L_{t+h} - L_t| \geq \epsilon) = 0$.*

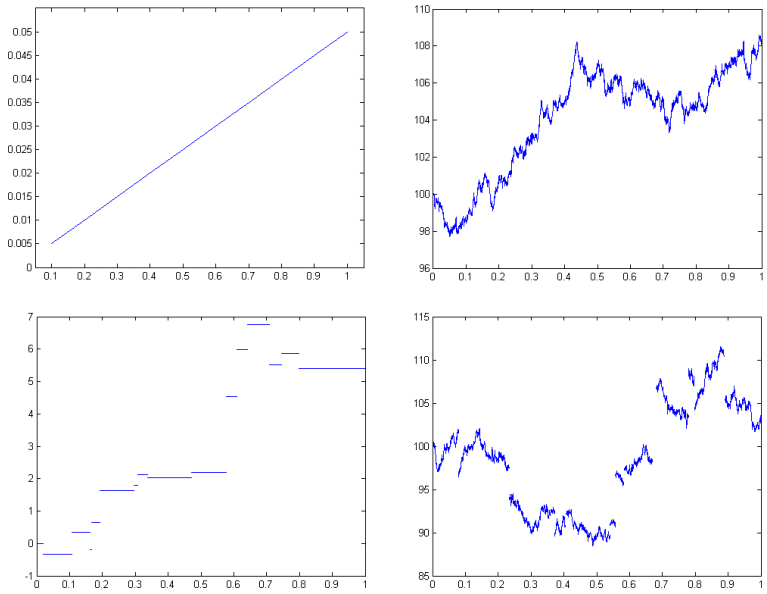


Figure 3.4: Examples of Lévy processes. A linear drift, a Poisson process, a Brownian motion and a Lévy jump-diffusion.

The third condition does not imply that the sample paths are continuous. It serves to exclude processes with jumps at fixed (nonrandom) times. It means that for given $t$, the probability of seeing a jump at $t$ is zero: discontinuities occur at random times.

The simplest Lévy process is the linear drift, a deterministic process. Brownian motion is the only (non-deterministic) Lévy process with continuous sample paths. Other examples of Lévy processes are the Poisson and compound Poisson processes. Notice that the sum of a Brownian motion and a compound Poisson process is again a Lévy process, often called a "jump-diffusion" process (see Figure 3.4).

### 3.2.1 Infinite Divisible Distributions

There is strong interplay between Lévy processes and infinitely divisible distributions.

**Definition 3.5 (Infinite divisibility)** *The law of a random variable $X$ is* infinitely divisible*, if for all $n \in \mathbb{N}$ there exist i.i.d. random variables $X_1^{(1/n)}, \ldots, X_n^{(1/n)}$ such that*

$$X \stackrel{d}{=} X_1^{(1/n)} + \ldots + X_n^{(1/n)}. \tag{3.16}$$

*Equivalently, the law $P_X$ of a random variable $X$ is* infinitely divisible *if for all $n \in \mathbb{N}$ there exists another law $P_{X^{(1/n)}}$ such that*

$$P_X = \underbrace{P_{X^{(1/n)}} * \ldots * P_{X^{(1/n)}}}_{n \quad times} \tag{3.17}$$

Alternatively, we can characterize an infinitely divisible random variable using its characteristic function.

**Definition 3.6** *The law of a random variable $X$ is* infinitely divisible*, if for all $n \in \mathbb{N}$, there exists a random variable $X^{(1/n)}$ such that*

$$\Phi_X(u) = \left(\Phi_{X^{(1/n)}}(u)\right)^n. \tag{3.18}$$

**Example 3.7 (Normal distribution)** *If $X \sim N(\mu, \sigma^2)$ then one can write $X = \sum_{k=0}^{n-1} Y_k^{(1/n)}$ where $Y_k^{(1/n)}$ are i.i.d. with law $N(\frac{\mu}{n}, \frac{\sigma^2}{n})$. The Normal distribution is thus infinitely divisible.*

*We can see that from the characteristic function:*

$$\begin{aligned}
\Phi_X(u) &= exp\left\{iu\mu - \frac{1}{2}u^2\sigma^2\right\} \\
&= exp\left\{iun\frac{\mu}{n} - \frac{1}{2}u^2 n\frac{\sigma^2}{n}\right\} \\
&= exp\left\{n\left(iu\frac{\mu}{n} - \frac{1}{2}u^2\frac{\sigma^2}{n}\right)\right\} \\
&= \left(exp\left\{iu\frac{\mu}{n} - \frac{1}{2}u^2\frac{\sigma^2}{n}\right\}\right)^n \\
&= (\Phi_{X^{(1/n)}}(u))^n
\end{aligned} \tag{3.19}$$

**Example 3.8 (Poisson distribution)** *Following the same procedure, we can easily deduce that the Poisson distribution is infinitely divisible. Let* $X \sim Poisson(\lambda)$*, the we have:*

$$\begin{aligned}
\Phi_X(u) &= exp\left\{\lambda(e^{iu} - 1)\right\} \\
&= exp\left\{n\frac{\lambda}{n}(e^{iu} - 1)\right\} \\
&= \left(exp\left\{\frac{\lambda}{n}(e^{iu} - 1)\right\}\right)^n \\
&= (\Phi_{X^{(1/n)}}(u))^n
\end{aligned} \tag{3.20}$$

*where* $X^{(1/n)} \sim Poisson(\frac{\lambda}{n})$*.*

### 3.2.2 The Lévy-Khintchine Formula

The next theorem provides a complete characterization of random variables with infinitely divisible distributions via their characteristic functions; this is the famous *Lévy-Khintchine formula*.

**Theorem 3.9 (Lévy-Khintchine formula)** *Let* $F$ *be an infinitely divisible*

*distribution on* $\mathbb{R}$*. Its characteristic function can be represented as:*

$$\Phi_F(u) = e^{\psi(u)} \qquad u \in \mathbb{R} \tag{3.21}$$

*where*

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (e^{iux} - 1 - iux\mathbf{1}_{|x|\leq 1})\nu(dx), \tag{3.22}$$

*where* $\gamma \in \mathbb{R}$*,* $\sigma \in \mathbb{R}_+$ *and* $\nu$ *is a positive measure satisfying:*

$$\int_{-1}^{+1} x^2\nu(dx) < \infty \qquad \int_{|x|\geq 1} \nu(dx) < \infty$$

$\nu$ *is called the Lévy measure of the distribution* $F$*.*

The triplet $(\gamma, \sigma^2, \nu)$ is called the *Lévy triplet*. The exponent $\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty}(e^{iux} - 1 - iux\mathbf{1}_{|x|\leq 1})\nu(dx)$ is called the *Lévy exponent*. Moreover, $\gamma \in \mathbb{R}$ is called the *drift term*, $\sigma \in \mathbb{R}_+$ the *Gaussian* or *diffusion coefficient* and $\nu$ the *Lévy measure*.

Now, consider a Lévy process $L = (L_t)_{t\geq 0}$; using the fact that, for any $n \in \mathbb{N}$ and any $t > 0$,

$$L_t = L_{t/n} + (L_{2t/n} - L_{t/n}) + \ldots + (L_t - L_{(n-1)t/n}) \tag{3.23}$$

together with the stationary and independence of the increments, we conclude that the *random variable* $L_t$ is infinitely divisible.

Then, for every Lévy process, the following holds

$$\begin{aligned} \mathbf{E}[e^{iuL_t}] &= e^{t\psi(u)} \\ &= \exp\left\{t\left(i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty}(e^{iux} - 1 - iux\mathbf{1}_{|x|\leq 1})\nu(dx)\right)\right\} \end{aligned}$$

where $\psi(u) := \psi_1(u)$ is the Lévy exponent of $L_1 := X$, a random variable with an infinitely divisible distribution.

We have seen so far, that every Lévy process can be associated with the law of an infinitely divisible distribution. The opposite, i.e. that given any random variable $X$, whose law is infinitely divisible, we can construct a Lévy process $L = (L_t)_{t\geq 1}$ such that $L_1 := X$, is also true. The law of $L_t$ is therefore determined by the knowledge of the law of $L_1$: the only degree of freedom we have in specifying a Lévy process is to specify the distribution of $L_1$ for a single time (say, $t = 1$).

### 3.2.3 The Lévy-Itô Decomposition

**Theorem 3.10 (Lévy-Itô decomposition)** *Consider a Lévy triplet $(\gamma, \sigma^2, \nu)$ where $\gamma \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$ and $\nu$ is a measure satisfying $\int_{-1}^{+1} x^2\nu(dx) < \infty$ and $\inf_{|x|\geq 1} \nu(dx) < \infty$. Then, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which four independent Lévy processes exist, $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ and $L^{(4)}$, where $L^{(1)}$ is a constant drift, $L^{(2)}$ is a Brownian motion, $L^{(3)}$ is a compound Poisson process and $L^{(4)}$ is a square integrable (pure jump) martingale with a countable number of jumps on each finite tome interval of magnitude less than 1. Taking $L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)}$, we have that there exists a probability space on which a Lévy process $L = (L_t)_{t\geq 0}$ with characteristic exponent*

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (e^{iux} - 1 - iux\mathbf{1}_{|x|\leq 1})\nu(dx) \tag{3.24}$$

*for all $u \in \mathbb{R}$, is defined.*

We can thus split the Lévy exponent into four parts:

$$\psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)} + \psi^{(4)} \tag{3.25}$$

where

$$\psi^{(1)}(u) = i\gamma u, \qquad \psi^{(2)} = \frac{1}{2}\sigma^2 u^2, \tag{3.26}$$

$$\psi^{(3)} = \int_{|x|\geq 1} (e^{iux} - 1)\nu(dx), \tag{3.27}$$

$$\psi^{(4)} = \int_{|x|<1} (e^{iux} - 1 - iux)\nu(dx), \tag{3.28}$$

Therefore, we can decompose any Lévy process into four independent Lévy processes $L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)}$, i.e.

$$\begin{aligned} L_t &= \gamma t + \sigma W_t + \int_0^t \int_{|x|\geq 1} x\mu^L(ds, dx) \\ &+ \left( \int_0^t \int_{|x|<1} x\mu^L(ds, dx) - t \int_{|x|<1} x\nu(dx) \right) \end{aligned} \tag{3.29}$$

where $L^{(1)}$ is a constant drift, $L^{(2)}$ is a Brownian motion, $L^{(3)}$ a compound Poisson process and $L^{(4)}$ is a pure jump martingale. This result is the celebrated *Lévy-Itô decomposition* of a Lévy process. ($\mu^L$ denotes the random measure counting the jumps of $L^{(4)}$).

The Lévy process $L$ can thus be interpreted as the independent superposition of a Brownian motion with drift and an infinite superposition of independent (compensated) Poisson processes with various jump sizes $x$, $\nu(dx)$ being the intensity of jumps of size $x$.

### 3.2.4 The Lévy measure

**Definition 3.11 (Lévy measure)** *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$. The measure $\nu$ on $\mathbb{R}$ defined by:*

$$\nu(A) = \mathbf{E}[\#\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}], \tag{3.30}$$

*is called the* Lévy measure *of $X$: $\nu(A)$ is the expected number, per unit time, of jumps whose size belongs to $A$.*

For example, the Lévy measure of the compound Poisson process is $\nu(dx) = \lambda F(dx)$; from that we can deduce that the expected number of jumps, in a time interval of length 1, is $\lambda$ and the jump size is distributed according to $F$. More generally, if $\nu$ is a finite measure, i.e. $\nu(\mathbb{R}) = \int_{\mathbb{R}} \nu(dx) = \lambda < \infty$, then $F(dx) := \frac{\nu(dx)}{\lambda} = 1$, which is a probability measure. Thus, $\lambda$ is the expected number of jumps and $F(dx)$ the distribution of the jump size $x$. If $\nu(\mathbb{R}) = \infty$, then an infinite number of (small) jumps is expected.

### 3.2.5 Lévy Processes as Martingales

The notion of martingale is crucial for probability theory and mathematical finance. Different martingales can be constructed from Lévy processes using their independent increments property.

**Proposition 3.12** *Let $L = (L_t)_{t\geq 0}$ be a Lévy process with Lévy exponent $\psi$ and assume $\mathbf{E}[e^{iuL_t}] < \infty$, $u \in \mathbb{R}$. The process $M = (M_t)_{t\geq 0}$, defined as*

$$M_t = \frac{e^{iuL_t}}{\mathbf{E}[e^{iuL_t}]} = \frac{e^{iuL_t}}{e^{t\psi(u)}} \tag{3.31}$$

*is a martingale.*

**Proposition 3.13** *Let $L = (L_t)_{t\geq 0}$ be a Lévy process with Lévy triplet $(\gamma, \sigma^2, \nu)$ and assume that $\mathbf{E}|L_t| < \infty$. $L$ is a martingale if and only if $\gamma = 0$.*

### 3.2.6 Path Properties

As we have seen earlier, the Lévy measure gives us the expected number of jumps of a certain height in a time interval of length 1. Hence, if $\nu$ is an infinite measure, then an infinite number of jumps is expected. Path properties of a Lévy process can be read from the Lévy measure. Processes which have a finite number of jumps on every time interval are called *finite activity* Lévy processes. Otherwise, processes which have an infinite one are called *infinite activity* Lévy processes.

**Proposition 3.14 (Finite/Infinite activity)** *Let $L$ be a Lévy process with triplet $(\gamma, \sigma^2, \nu)$.*

*(i) If $\nu(\mathbb{R}) < \infty$ then almost all paths of $L$ have a finite number of jumps on every compact interval. In that case, the Lévy process has* finite activity.

*(ii) If $\nu(\mathbb{R}) = \infty$ the almost all paths of $L$ have an infinite number of jumps on every compact interval. In that case, the Lévy process has* infinite activity.

**Proposition 3.15 (Finite/Infinite variation)** *Let $L$ be a Lévy process with triplet $(\gamma, \sigma^2, \nu)$.*

*(i) If $\sigma = 0$ (no Gaussian part) and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$ then almost all paths of $L$ have* finite variation.

*(ii) If* $\sigma \neq 0$ *or* $\int_{|x|\leq 1} |x|\nu(dx) = \infty$ *then almost all paths of* $L$ *have* infinite variation.

## 3.3 Stochastic Calculus for Processes with Jumps

Previously, we saw that the following formula

$$df = \left(\frac{\partial f}{\partial t} + b\frac{\partial f}{\partial x} + \sigma^2 \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right) dt + \frac{\partial f}{\partial x}\sigma dW_t \tag{3.32}$$

called Itô's Lemma for Wiener process, was a key tool which permits to describe the time evolution of a derivative instrument whose value $V_t = f(t, S_t)$ depends on $S_t$, when the model is a diffusion with Brownian motion. In fact, Itô's Lemma relates the local behavior of $V_t$ to the behavior of $S_t$. However, this version of Itô's Lemma cannot be use when the Wiener process is replaced by a process with jumps. We must, thus, define a new version of the Itô formula for jump processes.

### 3.3.1 Itô Formula for Jump processes

**Theorem 3.16 (Itô's Lemma for Jump-Diffusion Processes)** *Let* $X$ *be a diffusion process with jumps, defined as the sum of a drift term, a Brownian stochastic integral and a compound Poisson process:*

$$dX_t = bdt + \sigma dW_t + dJ_t, \tag{3.33}$$

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N_t} \Delta X_i, \tag{3.34}$$

*where* $b_t$ *and* $\sigma_t$ *are continuous nonanticiping processes with* $\mathbf{E}\left[\int_0^T \sigma_t^2 dt\right] < \infty$. *Then, for any* $C^{1,2}$ *function* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, *the process* $f(t, X_t)$ *can be*

*represented as:*

$$\begin{aligned} f(t,X_t) &= f(0,X_0) + \int_0^t \left(\frac{\partial f}{\partial s} + b_s\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_s^2\frac{\partial^2 f}{\partial x^2}\right) ds + \int_0^t \sigma_s \frac{\partial f}{\partial x} dW_s \\ &+ \sum_{i\geq 1, T_i \leq t} [f(X_{T_i} + \Delta X_i) - f(X_{T_i})], \end{aligned} \tag{3.35}$$

*or in differential notation*

$$df = \left(\frac{\partial f}{\partial t} + b_t\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2\frac{\partial^2 f}{\partial x^2}\right) dt \; + \; \frac{\partial f}{\partial x}\sigma_t dW_t \; + \; [f(X_{t-} + \Delta X_t) - f(X_{t-})]. \tag{3.36}$$

Itô's Lemma for jump-diffusion processes is similar to those for diffusion processes. It's actually the same with an expression more $[f(X_{t-}+\Delta X_t)-f(X_{t-})]$ which characterizes the jumps.

However, in the infinite activity case, an infinite number of jumps may occur in each interval. The number of jumps terms in the sum of the latter expression becomes infinite, and one cannot even figure out which are the effects on the evolution due to jumps from the one due to the Brownian component.

**Theorem 3.17 (Itô's Lemma for Lévy processes)** *Let $(X_t)_{t\geq 0}$ be a Lévy process with Lévy triplet $(\sigma^2, \nu, \gamma)$ and a $C^{1,2}$ function $f : \mathbb{R} \to \mathbb{R}$. The process $f(t, X_t)$ can be represented as:*

$$\begin{aligned} f(t,X_t) &= f(0,X_0) + \int_0^t \frac{\partial f}{\partial x}(s, X_{s-}) dX_s \\ &+ \int_0^t \left(\frac{\partial f}{\partial s}(s,X_s) + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(s,X_s)\right) ds \\ &+ \sum_{\substack{0\leq s<t \\ \Delta X_s \neq 0}} [f(s, X_{s-} + \Delta X_s) - f(s, X_{s-}) - \Delta X_s \frac{\partial f}{\partial x}(s, X_{s-})] \end{aligned} \tag{3.37}$$

*or in differential notation*

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}dX_t + [f(X_{t-}+\Delta X_t) - f(X_{t-}) - \Delta X_t \frac{\partial f}{\partial x}(X_{t-})] \tag{3.38}$$

### 3.3.2 Equivalence of Measures for Lévy Processes

As the Girsanov theorem in a diffusion contest, we present a method to convert the discounted underlying price $e^{-rt}S_t$ into a martingale. We are thus looking for a measure $\mathbb{Q}$ such that

$$\mathbf{E}^{\mathbb{Q}}[\hat{S}_T|\mathcal{F}_t] = \hat{S}_t, \qquad where \qquad \hat{S}_t = e^{-rt}S_t \tag{3.39}$$

There exists a method called the Esscher transform which permits to obtain the equivalent martingale measure $\mathbb{Q}$ for a Lévy process by an exponential transform. However, this method remains a bit complex and the measure obtained doesn't seem to be the kind of measure chosen by the market. Another way to obtain an equivalent martingale measure $\mathbb{Q}$ is by *mean-correcting* the exponential of a Lévy process. This method proposes to change the drift (or to add a drift if the process doesn't have a drift part, see remark below) in order to make the discounted stock price become a martingale.

**Remark 3.18** *For processes without drift part (as the VG or NIG, see next chapter), an additional drift $p \in \mathbb{R}$ can be introduced without altering the infinite divisibility property nor the self-decomposability of the distribution.*
*The new distribution has a characteristic function $\hat{\Phi}$ in terms of the original characteristic function $\Phi$*

$$\hat{\Phi}(u) = \Phi(u)e^{iup} \tag{3.40}$$

*The drift parameter just translate the distribution by the value p. Thus, a term pt is added to the process* $L_t$

$$\hat{L}_t = L_t + pt \tag{3.41}$$

*Finally, the first parameter of the Lévy triplet is the only changed:* $(\gamma{+}p, \sigma^2, \nu)$.

Let's explain the *mean-correcting martingale measure method.*
First of all, we set $S_0 = 1$ for simplicity of calculation. Then, strike price $K$ is given in function of $S_0$ (for example, a strike price of 0.9 means 90% of $S_0$; if $S_0 = 100$, the strike price is $K = 90$).
We know that the characteristic function of a Lévy process is of the form $e^{\psi(u)}$ where $\psi(u)$ is called the characteristic exponent.
We know that for a Lévy process, the characteristic exponent is of the form

$$\psi(u) = t\left(i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty}(e^{iux} - 1 - iux\mathbf{1}_{|x|\leq 1})\nu(dx)\right) \tag{3.42}$$

We can thus divide the characteristic exponent into two parts: a drift part and another no-drift part

$$\psi(u) = \mu(u) + \varphi(u) \tag{3.43}$$

where

$$\varphi(u) = t\left(-\frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty}(e^{iux} - 1 - iux\mathbf{1}_{|x|\leq 1})\nu(dx)\right) \tag{3.44}$$

and

$$\mu(u) = i\gamma ut \tag{3.45}$$

(If the process doesn't have a drift part, just set $\mu(u) = 0$; moreover, the Gaussian part $-\frac{1}{2}\sigma^2 u^2 t$ or the jump part $t\int_{-\infty}^{+\infty}(e^{iux} - 1 - iux\mathbf{1}_{|x|\leq 1})\nu(dx)$ can

also be 0).

It can be shown that under the risk-neutral measure, the (new) drift part $\mu^{RN}(u)$ is given by

$$\mu^{RN}(u) = i\left(r - \frac{\varphi(-i)}{t}\right)ut \tag{3.46}$$

where $r$ is the risk-free interest rate and $\Delta = r - \frac{\varphi(-i)}{t}$ is the drift parameter under the risk-neutral measure.

Then, the risk-neutral characteristic function is given by

$$\begin{aligned}\Phi_{RN}(u) &= e^{\varphi(u)}e^{\mu^{RN}(u)} \\ &= e^{\varphi(u)+\mu^{RN}(u)} \\ &= e^{\varphi(u)+i\Delta ut}\end{aligned} \tag{3.47}$$

Therefore, under the risk-neutral measure, the characteristic exponent is of the form

$$\varphi(u) = t\left\{\left(r - \frac{\varphi(-i)}{t}\right)iu - \frac{1}{2}\sigma^2u^2 + \int_{-\infty}^{+\infty}(e^{iux} - 1 - iux\mathbf{1}_{|x|\leq 1})\nu(dx)\right\} \tag{3.48}$$

where, as seen before, the Gaussian part $-\frac{1}{2}\sigma^2u^2t$ or the jump part $t\int_{-\infty}^{+\infty}(e^{iux} - 1 - iux\mathbf{1}_{|x|\leq 1})\nu(dx)$ can be 0.

**Example 3.19 (Black-Scholes)** *The characteristic function of the Black-Scholes model under the measure $\mathbb{P}$ is given by:*

$$\Phi(u) = \exp\left\{t\left(i\gamma u - \frac{1}{2}\sigma^2u^2\right)\right\} \tag{3.49}$$

*We can then split the characteristic exponent into two parts, a drift part $\mu(u) = i\gamma ut$ and a no-drift part $\varphi(u) = -\frac{1}{2}\sigma^2u^2t$.*

*Therefore, under the risk-neutral measure, the drift part can be written as*

$$\mu^{RN}(u) = i\left(r - \frac{\varphi(-i)}{t}\right)ut = i\left(r - \frac{1}{2}\sigma^2\right)ut \tag{3.50}$$

*So, the risk-neutral characteristic function of the Black-Scholes model is given by*

$$\begin{aligned}\Phi_{RN}(u) &= \exp\left\{-\frac{1}{2}\sigma^2u^2t\right\}\exp\left\{i\left(r - \frac{1}{2}\sigma^2\right)ut\right\} \\ &= \exp\left\{t\left(i\Delta u - \frac{1}{2}\sigma^2u^2\right)\right\}\end{aligned} \tag{3.51}$$

*where $\Delta$ is the risk-neutral drift parameter*

$$\Delta = r - \frac{1}{2}\sigma^2 \tag{3.52}$$

In the next chapter, thanks to this method, we will derive the risk-neutral characteristic functions for various Lévy processes.

# Chapter 4

# Lévy Processes for Financial Modeling

Lévy processes useful in finance fall into two categories. In the first category, called *jump-diffusion* models, the "normal" evolution of prices is given by a diffusion process, punctuated by jumps at random intervals. Hence, the jumps represent rare events. Such a model can be modeled by a Gaussian part (Brownian motion) plus a jump part, which is compound Poisson process with finitely many jumps in every time interval and distribution $F$ of jump size. On figure 4.1, we can clearly see the jumps which characterize the jump-diffusion model. Such models are easy to simulate and lead to good fit of the volatility smile in the long run. However, they rarely lead to closed-form densities.

The second category consists of models with infinite number of jumps in every time interval, called *infinite activity* models. In these models, one does not need to introduce a Brownian component since the dynamics of jumps are already rich enough to generate nontrivial small time behavior; these models

are *infinite activity pure jump models*. However, many models of this type can be constructed by Brownian subordination. We could also consider pure jump processes of finite activity without diffusion component but these models do not lead to a realistic description of price dynamics.

All the models we are about to discuss belong to a family of Lévy processes called "exponential Lévy processes". In this class of Lévy processes, the risk-neutral dynamics of the underlying asset is given by

$$S_t = S_0 \exp(L_t) \tag{4.1}$$

where $L_t$ is a Lévy process under the equivalent martingale measure $\mathbb{Q}$ with characteristic triplet $(\gamma, \sigma^2, \nu)$. The log returns $\log(S_{t+s}/S_t)$ of such a model follow the distribution of increments of length $s$ of the Lévy process $L_t$. We choose $L_0 = 0$ to have $\exp(L_0) = 1$ and thus $S_0 = S_0 \exp(L_0) = S_0$. If we set $S_0 = 1$, the process can be resumed as $S_t = \exp(L_t)$.
The absence of arbitrage then imposes that $\hat{S}_t = S_t e^{-rt} = e^{-rt} e^{L_t}$ be a martingale.
As seen in section 3.3.2, we are able to find an equivalent martingale measure under which the discounted stock price $\hat{S}_t$ is a martingale (i.e. we are able to derive the risk-neutral characteristic function for the discounted Lévy process). Different exponential Lévy models proposed in the financial modeling literature simply correspond to different choices for the Lévy measure $\nu$ (and for the Gaussian part $\sigma$ if present).
The Black-Scholes model is thus nothing more than an exponential Lévy model

where the Lévy process $L_t$ is a Brownian motion with drift $\gamma = r - \frac{1}{2}\sigma^2$

$$\begin{aligned} L_t &= \gamma t + \sigma W_t \\ &= (r - \frac{1}{2}\sigma^2)t + \sigma W_t \end{aligned} \tag{4.2}$$

then,

$$\begin{aligned} S_t &= S_0 \exp(L_t) \\ &= S_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W_t\right) \end{aligned} \tag{4.3}$$

But as $L_{t_i} = L_{t_{i-1}} + \gamma(t_i - t_{i-1}) + \sigma\sqrt{t_i - t_{i-1}}N(0,1)$, we can rewrite the last equation as

$$\begin{aligned} S_{t_i} &= S_0 \exp(L_{t_i}) \\ &= S_0 \exp\left\{L_{t_{i-1}} + (r - \frac{1}{2}\sigma^2)(t_i - t_{i-1}) + \sigma\sqrt{t_i - t_{i-1}}N(0,1)\right\} \\ &= S_0 \exp(L_{t_{i-1}}) \exp\left\{(r - \frac{1}{2}\sigma^2)(t_i - t_{i-1}) + \sigma\sqrt{t_i - t_{i-1}}N(0,1)\right\} \\ S_{t_i} &= S_{t_{i-1}} \exp\left\{(r - \frac{1}{2}\sigma^2)(t_i - t_{i-1}) + \sigma\sqrt{t_i - t_{i-1}}N(0,1)\right\} \end{aligned} \tag{4.4}$$

Before going ahead, we need to define a Lévy process called *subordinator* which has an important role in the construction of other Lévy processes.

**Definition 4.1 (Subordinator)** *A* subordinator *is an increasing (in t) Lévy process. Equivalently, for S to be a subordinator, the triplet must satisfy* $\nu(-\infty, 0) = 0$, $c = 0$, $\int_{(0,1)} x\nu(dx) < \infty$ *and* $\gamma = b + \int_{(0,1)} x\nu(dx) > 0$.

This class of Lévy processes is called subordinator because they can be used as time changes for other Lévy processes. They are very important ingredients

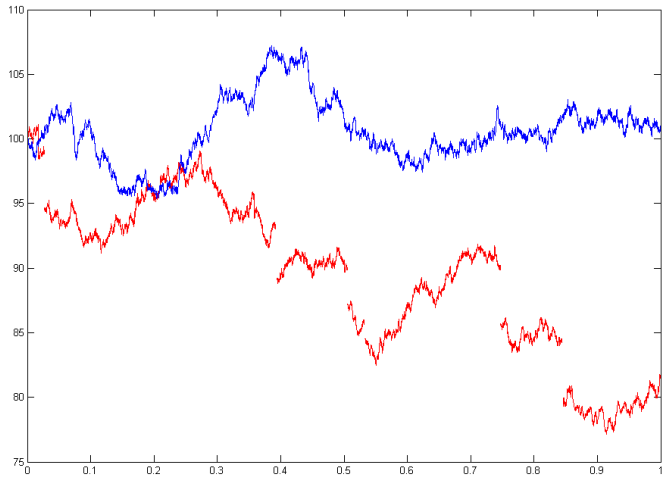



Figure 4.1: Paths of the Black-Scholes model (in blue) and the Merton jump-diffusion model (in red). We used $r = 5\%$, $\sigma = 20\%$, $\lambda = 5$, $\alpha = -0.1$ and $\delta = 0.4$

for building Lévy-based models in finance. The Poisson, the Gamma and the inverse Gaussian process are examples of subordinator.

## 4.1 Jump-Diffusion Models

Assume that the process $L = (L_t)_{t\geq 0}$ is a Lévy jump-diffusion, i.e. a Brownian motion plus a compensated compound Poisson process. It can be described by

$$L_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i \tag{4.5}$$

where $\gamma \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, $W = (W_t)_{t\geq 0}$ is a standard Brownian motion, $N = (N_t)_{t\geq 0}$ is a Poisson process with parameter $\lambda$ (i.e. $\mathbf{E}[N_t] = \lambda t$) and $Y = (Y_t)_{t\geq 1}$ is an i.i.d. sequence of random variables with probability distribution $F$ and

$\mathbf{E}[Y] = k < \infty$; $F$ describes the distribution of jump size. All sources of randomness are mutually independent.

The characteristic function of $L_t$ is

$$\begin{aligned} \mathbf{E}[e^{iuL_t}] &= \mathbf{E}\left[\exp\left\{iu\left(\gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right)\right\}\right] \\ &= \exp\{iu\gamma t\}\ \mathbf{E}\left[\exp\{iu\sigma W_t\}\ \exp\left\{iu\sum_{i=1}^{N_t} Y_i\right\}\right] \end{aligned} \tag{4.6}$$

since all the sources of randomness are independent, we get

$$= \exp\{iu\gamma t\}\ \mathbf{E}\left\{\exp(iu\sigma W_t)\right\}\ \mathbf{E}\left[\exp\left\{iu\sum_{i=1}^{N_t} Y_i\right\}\right]$$

taking into account that

$$\mathbf{E}[e^{iuW_t}] = e^{-\frac{1}{2}\sigma^2u^2t}, \quad W_t \sim Normal(0, \sigma^2 t) \tag{4.7}$$

$$\mathbf{E}[e^{iu\sum_{i=1}^{N_t} Y_i}] = e^{\lambda t(\mathbf{E}[e^{iuY}-1])}, \quad N_t \sim Poisson(\lambda t) \tag{4.8}$$

we get

$$= \exp\{iu\gamma t\}\exp\left\{-\frac{1}{2}u^2\sigma^2 t\right\}\exp\left\{\lambda t\left(\mathbf{E}\left[e^{iuY}-1\right]\right)\right\} \tag{4.9}$$

and because the distribution of $Y$ is $F$ we have

$$= \exp\{iu\gamma t\}\exp\left\{-\frac{1}{2}u^2\sigma^2 t\right\}\exp\left\{\lambda t\int_{-\infty}^{+\infty}(e^{iux}-1)F(dx)\right\} \tag{4.10}$$

Now, since $t$ is a common factor, we write the above equation as

$$\mathbf{E}[e^{iuL_t}] = \exp\left\{t\left(iu\gamma - \frac{1}{2}u^2\sigma^2 + \int_{-\infty}^{+\infty}(e^{iux}-1)\lambda F(dx)\right)\right\} \tag{4.11}$$

Since the characteristic function of a random variable determines its distribution, we have a "characterization" of the distribution of the random variables underlying the Lévy jump-diffusion.

### 4.1.1 Merton Model

Merton (1976) was the first to use a discontinuous price process to model asset returns. In the *Merton model*, jumps in the log-price $X_t$ are assumed to have a Gaussian distribution. The canonical decomposition of the driving process is

$$X_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} Y_i \tag{4.12}$$

where $Y_i \sim N(\alpha, \delta^2)$. Hence, the distribution of the jump size has the density

$$F(x) = \frac{1}{\delta\sqrt{2\pi}} \exp\left[-\frac{(x-\alpha)^2}{2\delta^2}\right] \tag{4.13}$$

and the Lévy density

$$\nu(x) = \lambda F = \frac{\lambda}{\delta\sqrt{2\pi}} \exp\left[-\frac{(x-\alpha)^2}{2\delta^2}\right] \tag{4.14}$$

The *Merton model* has thus 4 parameters (excluding the drift $\mu$): $\sigma$ the diffusion volatility, $\lambda$ the jump intensity, $\alpha$ the mean jump size and $\delta$ the standard deviation of jump size.

The characteristic function of the model is given by

$$\mathbf{E}[e^{iuX_t}] = \exp\left\{t\left(i\mu u - \frac{1}{2}\sigma^2 u^2 + \lambda(e^{i\alpha u - \frac{1}{2}\delta^2 u^2} - 1)\right)\right\} \tag{4.15}$$

and the Lévy triplet by $(\mu, \sigma^2, \lambda F)$.

The density of $X_t$ is not none in closed form, but it admits a series expansion

$$p_t(x) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k \exp\left\{-\frac{(x-\mu t-\alpha k)^2}{2(\sigma^2 t+\delta^2 k)}\right\}}{k!\sqrt{2\pi(\sigma^2 t + \delta^2 k)}} \tag{4.16}$$

and the first two moments are

$$\mathbf{E}[X_t] = t(\mu + \lambda\delta) \qquad \mathbf{Var}[X_t] = t(\sigma^2 + \lambda\alpha^2 + \lambda\delta^2)$$

### 4.1.2 Kou Model

Kou (2002) proposed a jump-diffusion similar Merton's, where the jump size is double-exponentially distributed. Therefore, the canonical decomposition of the driving process of the *Kou model* is

$$X_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} Y_i \tag{4.17}$$

where $Y_i \sim DbExpo(p, \eta_1, \eta_2)$. Hence, the distribution of the jump size has the density

$$F(x) = p\eta_1 e^{-\eta_1 x}\mathbf{1}_{x<0} + (1-p)\eta_2 e^{-\eta_2|x|}\mathbf{1}_{x>0} \tag{4.18}$$

and the Lévy density

$$\nu(x) = \lambda F = p\lambda\eta_1 e^{-\eta_1 x}\mathbf{1}_{x<0} + (1-p)\lambda\eta_2 e^{-\eta_2|x|}\mathbf{1}_{x>0} \tag{4.19}$$

The *Kou model* has thus 5 parameters (excluding the drift $\mu$): $\sigma$ the diffusion volatility, $\lambda$ the jump intensity, $p$ the probability of an upward jump and $\eta_1$ and $\eta_2$ govern the decay of the tails for the distribution of positive and negative jump sizes.

The characteristic function of the model is given by

$$\mathbf{E}[e^{iuX_t}] = \exp\left\{t\left(i\mu u - \frac{1}{2}\sigma^2 u^2 + iu\lambda\left[\frac{p}{\eta_1 - iu} - \frac{1-p}{\eta_2 + iu}\right]\right)\right\} \tag{4.20}$$

and the Lévy triplet by $(\mu, \sigma^2, \lambda F)$ The density of $X_t$ is not known in closed form, while the first two moments are

$$\mathbf{E}[X_t] = t\left(\mu + \frac{\lambda p}{\eta_1} - \frac{\lambda(1-p)}{\eta_2}\right) \qquad \mathbf{Var}[X_t] = t\left(\sigma^2 + \frac{\lambda p}{\eta_1^2} - \frac{\lambda(1-p)}{\eta_2^2}\right)$$

## 4.2 Infinite Activity Models

During the last decade, a vast amount of papers has been written about Lévy processes and infinite activity models have clearly represented the major area of research. Infinite activity models form a rich class of processes and the study of this entire family is far beyond the scope of this thesis. We will just present two models: the Variance Gamma (VG) model and the Normal Inverse Gaussian (NIG) model. The VG model was introduced in 1987 by Madan and Seneta [71] as a model for stock returns. They considered the symmetric case with $\theta = 0$ (see also the articles from Madan and Seneta [72] and from Madan and Milne [70]). The general case has been studied in 1998 in the paper from Madan, Carr and Chang [69]. The NIG distribution was introduced by Barndorff-Nielsen in 1995 [10]. See also [11], [12] and [84] for further development.

These two models are famous for their easy simulation. Indeed, they can be simulated through Brownian subordination. Subordinating Brownian motion with drift $\mu$ by the (subordinator) process $S$, we obtain a new Lévy process $L_t = \mu S_t + \sigma W_{S_t}$. This process is a Brownian motion if observed on a new time scale, that is, the stochastic time scale given by $S_t$ (we have replaced the deterministic time argument $t$ by a random time $S_t$. This scale has the financial interpretation of business time [52], that is, the integrated rate of information arrival. This interpretation makes models based on subordinated Brownian motion easier to understand that general Lévy models.

The subordinating processes under VG and NIG are respectively Gamma process and Inverse Gaussian process.

### 4.2.1 Normal Inverse Gaussian

We first present the subordinating Inverse Gaussian process and then talk about the Normal Inverse Gaussian process.

#### The Inverse Gaussian Process

The inverse Gaussian distribution with parameters $a, b > 0$ has density

$$f_{IG}(x) = \frac{ae^{ab}}{\sqrt{2\pi}} x^{-3/2} \exp(-\frac{1}{2}(a^2 x^{-1} + b^2 x)) \qquad x > 0. \tag{4.21}$$

This is the density of the first passage time to level $a$ of a Brownian motion with drift $b$. It has mean $a/b$ and variance $a/b^3$. The characteristic function of the Inverse-Gaussian process is given by

$$\Phi_{IG}(u; a, b) = \exp\left\{-a(\sqrt{-2iu + b^2} - b)\right\} \tag{4.22}$$

The inverse Gaussian distribution is infinitely divisible: if $X_1$ and $X_2$ are independent and have this density with parameters $(a_1, b)$ and $(a_2, b)$, then it is clear from the first passage time interpretation that $X_1 + X_2$ has this density with parameters $(a_1 + a_2, b)$. It follows that we can define the IG process $X^{(IG)} = \{X_t^{(IG)}, t \geq 0\}$, with parameters $a, b > 0$, as the process which starts at zero and has independent and stationary increments such that

$$\begin{aligned} \mathbf{E}[e^{iuX_t^{(IG)}}] &= \Phi_{IG}(u; at, b) \\ &= \exp\left\{-at(\sqrt{-2iu + b^2} - b)\right\} \end{aligned} \tag{4.23}$$

An IG random variable takes only positive values so an IG process is non-decreasing. This makes it unsuitable as a model of (the logarithm of) a risky asset price but perfect as a subordinator.

The IG distribution satisfies the following scaling property. IF $X$ is $IG(a, b)$, then, for a positive $c$

$$cX \sim IG(a\sqrt{c}, \frac{b}{\sqrt{c}}) \tag{4.24}$$

**The NIG Process**

The Normal Inverse Gaussian (NIG) distribution has a characteristic function given by

$$\Phi_t(u, \alpha, \beta, \delta) = \exp\left\{-\delta\left(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}\right)\right\} \tag{4.25}$$

This characteristic function is also infinitely divisible, so we can define the NIG process $X^{(NIG)} = \{X_t^{(NIG)}, t \geq 0\}$ with 3 parameters $\alpha : \alpha > 0$, $\beta : -\alpha < \beta < \alpha$ and $\delta : \delta > 0$, with $X_0^{(NIG)} = 0$ stationary and independent NIG distributed increments. The characteristic function of the NIG process is therefore

$$\begin{aligned} \mathbf{E}[e^{iuX_t^{(NIG)}}] &= \Phi_t^{(NIG)}(u; \alpha, \beta, \delta t) \\ &= \exp\left\{-\delta t\left(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}\right)\right\} \end{aligned} \tag{4.26}$$

The NIG process is a Lévy process without Gaussian part. It can can be called a pure jump process. Its path is rich enough (has an infinitely number of small jumps) to not need to introduce a Brownian part.

For its simulation (or for a better understanding of its path), we can relate the NIG process to an Inverse Gaussian time-changed Brownian motion, that is we can see it has a Brownian motion subordinated by an Inverse Gaussian process.

Let $W_t$ be a standard Brownian motion and let $I_t$ be an IG process with parameters $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$, then the stochastic process

$$X_t = \beta\delta^2 I_t + \delta W_{I_t} \tag{4.27}$$

is an NIG process with parameters $\alpha$, $\beta$ and $\delta$.
The first two moments of the NIG process are

$$\mathbf{E}[X_t] = \delta t \beta \sqrt{\alpha^2 - \beta^2} \qquad \mathbf{Var}[X_t] = \alpha^2 \delta t (\alpha^2 - \beta^2)^{-3/2}$$

### 4.2.2 Variance Gamma

We first present the subordinating Gamma process and then talk about the Variance Gamma process.

#### The Gamma Process

The Gamma distribution with parameters $a, b > 0$ has density

$$f_G(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-xb} \qquad x > 0 \tag{4.28}$$

The characteristic function is given by

$$\Phi_{Gamma}(u; a, b) = \left(1 - \frac{iu}{b}\right)^{-a} \tag{4.29}$$

The Gamma distribution has mean $a/b$ and variance $a/b^2$.
If $Y_1, \ldots, Y_n$ are independent with distribution Gamma$(a/n, b)$, then $Y_1 + \ldots + Y_n$ has distribution Gamma$(a, b)$; thus, Gamma distributions are infinitely divisible. For each choice of the parameters $a$ and $b$ there is a Lévy process (called a *Gamma* process) such that $X(1)$ has distribution Gamma$(a, b)$. Then, the Gamma process $X^{(Gamma)} = \{X_t^{(Gamma)}, t \geq 0\}$ is defined as the stochastic process which starts at zero and has stationary and independent Gamma distributed increments, $X_t^{(Gamma)}$ follows a Gamma$(at, b)$ distribution and as a

characteristic function given by

$$\begin{aligned}\mathbf{E}[e^{iuX_t^{(Gamma)}}] &= \Phi_t^{(Gamma)}(u; at, b) \\ &= \left(1 - \frac{iu}{b}\right)^{-at} \end{aligned} \tag{4.30}$$

A Gamma random variable takes only positive values so a Gamma process is non-decreasing. This makes it unsuitable as a model of (the logarithm of) a risky asset price but perfect as a subordinator.

The Gamma distribution satisfies the following scaling property. If $X$ is Gamma$(a, b)$, then for a positive $c$

$$cX \sim Gamma(a, \frac{b}{c}) \tag{4.31}$$

**The VG Process**

The characteristic function of the VG$(\sigma, \nu, \theta)$ law is given by

$$\Phi(u; \sigma, \nu, \theta) = \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-1/\nu} \tag{4.32}$$

The distribution is infinitely divisible and we can define a VG process $X_t^{(VG)}$ as the process which starts at zero, has independent and stationary increments and for which the increments $X_{s+t}^{(VG)} - X_s^{(VG)}$ follows a $VG(\sigma\sqrt{t}, \nu/t, t\theta)$ law over the time interval $[s, t+s]$.

The characteristic function of the VG process is thus

$$\begin{aligned}\mathbf{E}[e^{iuX_t^{(VG)}}] &= \Phi_{VG}(u; \sigma\sqrt{t}, \nu/t, t\theta) \\ &= (\Phi_{VG}(u; \sigma, \nu, \theta))^t \\ &= \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-t/\nu} \end{aligned} \tag{4.33}$$

Madan and Seneta [72] showed that a Variance Gamma process can be seen as a Lévy process $L_t$ with $L_t = G_t^{(1)} - G_t^{(2)}$, where $G^{(1)}$ and $G^{(2)}$ are independent Gamma processes representing the up and down moves of $L_t$.
We can rewrite the characteristic function as

$$\Phi_{VG}(u; C, G, M) = \left( \frac{GM}{GM + (M - G)iu + u^2} \right)^C \tag{4.34}$$

where

$$C = 1/\nu \qquad > 0 \tag{4.35}$$

$$G = \left( \sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu \right)^{-1} \qquad > 0 \tag{4.36}$$

$$M = \left( \sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\theta\nu \right)^{-1} \qquad > 0 \tag{4.37}$$

with this parameterization, we have thus

$$L_t^{(VG)} = G_t^{(1)} - G_t^{(2)} \tag{4.38}$$

where $G_t^{(1)}$ is a Gamma process with parameters $a = C$ and $b = M$, whereas $G_t^{(2)}$ is an independent Gamma process with parameters $a = C$ and $b = G$. We can refer to $\mathrm{VG}(C, G, M)$.

If $G_1^{(1)}$ and $G_1^{(2)}$ have the same shape and scale parameters, then $L_t$ admits an alternative representation as $W_{G_t}$ where $W$ is a standard Brownian motion and $G$ is a Gamma process. In other words, $L_t$ can be viewed as the result of applying a random time-change to an ordinary Brownian motion.
More precisely, let $G_t$ be a Gamma process with parameters $a = 1/\nu > 0$ and $b = 1/\nu > 0$. Let $W_t$ denote a standard Brownian motion, let $\sigma > 0$ and $\theta \in \mathbb{R}$; then the VG process $X_t^{(VG)}$, with parameters $\sigma > 0$, $\nu > 0$ and $\theta$, can

alternatively be defined as

$$X_t^{(VG)} = \theta G_t + \sigma W_{G_t} \tag{4.39}$$

If $\theta = 0$, then $G = M$ and the distribution is symmetric (the parameters $\theta$ controls the skewness). The parameters $\nu = 1/C$ controls the kurtosis.
The first two moments of the VG process are

$$\mathbf{E}[X_t] = \theta t \qquad \mathbf{Var}[X_t] = \sigma^2 t + \theta^2 \nu t \tag{4.40}$$

or with the CGM notations

$$\mathbf{E}[X_t] = \frac{Ct(G-M)}{MG} \qquad \mathbf{Var}[X_t] = \frac{Ct(G^2+M^2)}{(MG)^2} \tag{4.41}$$

Madan et al. [69] considered the more general case $W_{G_t}$ where $W$ now has the drift parameter $\mu$ and variance parameter $\sigma^2$. They restricted the shape parameter of $G_1$ to be the reciprocal of its scale parameter $b$ (so that $\mathbf{E}[G_t] = t$) and show that this more general Variance Gamma process can still be represented as the difference $G_t^{(1)} - G_t^{(2)}$ of two independent Gamma processes. The shape and scale parameters of $G_1^{(1)}(a_1, b_2)$ and $G_1^{(2)}(a_2, b_2)$ should be chosen to satisfy

$$\begin{aligned} a_1 &= 1/b \\ a_2 &= 1/b \\ b_1 b_2 &= (\sigma^2 b)/2 \\ b_1 - b_2 &= \mu b \end{aligned}$$

## 4.3 Simulation

In this section, we describe the techniques used to simulate the last four processes. The code written in MATLAB to realize these simulations can be found in the appendix at the end of this thesis and downloadable at the following website: *http://ddeville.110mb.com/thesis/*

As random variates generators (for Normal, Poisson, Inverse Gaussian and Gamma random numbers) are the basis of our simulations, we have to be sure of their goodness. Based upon the fact that

$$\frac{1}{N}\sum_{i=1}^{N} X_i \to \mathbf{E}[X] \qquad \frac{1}{N}\sum_{i=1}^{N} (X - \mathbf{E}[X])^2 \to \mathbf{Var}[X] \tag{4.42}$$

we decide to calculate the empirical mean and variance for $N = 10,000$ and check out how close they are to the theoretical ones. A deeper coverage of the subject can be found in the book by Devroye [40].

### 4.3.1 Jump Diffusion Processes

As many processes have a Brownian part (or a subordinated Brownian part), we need an efficient algorithm to generate Normal random variates. The most famous algorithm to generate Normal random variates is the Bow-Muller generator. It generates two independent Normal random variates.

**Algorithm 4.2 (Box-Muller generator of Normal variates)**

*Generate two uniform* $[0,1]$ *random variates* $U$, $V$.
*Set* $E \leftarrow -log(U)$ *(exponentially distributed)*
*Set* $A \leftarrow cos(2\pi V)\sqrt{2E}$

| Empirical mean | Theoretical mean |
|---|---|
| +2.0015 | +2.0000 |
| Empirical variance | Theoretical variance |
| +3.0014 | +3.0000 |

Table 4.1: Normal(2,3) with Box-Muller Normal variates generator.

*Set* $B \leftarrow sin(2\pi V)\sqrt{2E}$
*RETURN(A,B)*

Given its goodness and quickness we will use the Random Normal variates generator `randn` in MATLAB

### Poisson Process

To generate Jump-Diffusion processes, we need to add a jump part to a Brownian motion with drift. This jump part is represented by random jump times given by a Poisson process.
We present then three generators of Poisson random variates with a comparable goodness.

**Algorithm 4.3 (Poisson generator: exponential inter-arrival times)**

*Set* $X \leftarrow 0$
*Set* $Sum \leftarrow 0$
*WHILE true DO*
    *Generate an exponential random variate* $E$.
    $Sum \leftarrow Sum + E$
    *IF* $Sum < \lambda$
        *THEN* $X \leftarrow X + 1$

*ELSE RETURN X*

**Algorithm 4.4 (Poisson generator: multiplication of uniform r.v)**

*Set* $X \leftarrow 0$
*Set* $Prod \leftarrow 1$
*WHILE true DO*
    *Generate a uniform random variate* $U$.
    $Prod \leftarrow ProdU$
    *IF* $Prod > e^{-\lambda}$
        *THEN* $X \leftarrow X + 1$
        *ELSE RETURN X*

**Algorithm 4.5 (Poisson generator: inversion by sequential search)**

*Set* $X \leftarrow 0$
*Set* $Sum \leftarrow e^{-\lambda}$
*Set* $Prod \leftarrow e^{-\lambda}$
*Generate a uniform random variate* $U$.
*WHILE* $U > Sum$ *DO*
    $X \leftarrow X + 1$
    $Prod \leftarrow \frac{1}{X} Prod$
    $Sum \leftarrow Sum + Prod$
    *THEN* $X \leftarrow X + 1$
*RETURN X*

We choose to use the first generator even though the three have the same characteristics (just the second has Variance a little over-estimated).

| Empirical mean | Theoretical mean |
|---|---|
| +9.9666 | +10.0000 |
| +10.0235 | +10.0000 |
| +10.0860 | +10.0000 |
| Empirical variance | Theoretical variance |
| +10.0436 | +10.0000 |
| +10.1230 | +10.0000 |
| +9.9712 | +10.0000 |

Table 4.2: Poisson(10), First line: exponential inter-arrival times, Second line: multiplication of uniform r.v., Third line: inversion by sequential search.

Then we present an algorithm to generate a sample path of a Poisson process. The Poisson process is a no-decreasing process of increments 1 and exponential spacings between increments.

**Algorithm 4.6 (Sample path of a Poisson process)** *Simulation of the process $(X(t_1), \ldots, X(t_n))$ for fixed times $t_1, \ldots, t_n$): a discretized trajectory of the Poisson process with parameter $\lambda$.*

- *Simulate $n$ i.i.d. Poisson random variables $I_t$ with parameter $h\lambda$ ($h$ is the discretization step)*

*The discrete trajectory of the process is given by*

$$X_{t_i} = X_{t_{i-1}} + I_{t_i}$$

The Poisson process is not a martingale. Its counterpart, the compensated Poisson process, is.

### Compensated Poisson Process

**Algorithm 4.7 (Sample path of a compensated Poisson process)** *Simulation of the process $(X(t_1), \ldots, X(t_n))$ for fixed times $t_1, \ldots, t_n$): a discretized trajectory of the compensated Poisson process with parameter $\lambda$.*

- *Simulate $n$ i.i.d. Poisson random variables $I_t$ with parameter $h\lambda$ ($h$ is the discretization step)*

*The discrete trajectory of the process is given by*

$$X_{t_i} = X_{t_{i-1}} + I_{t_i} - \lambda(t_i - t_{i-1})$$

The compound Poisson process is a Poisson process which increments are not of size 1 anymore but given by a distribution $F$.

### Compound Poisson Process

**Algorithm 4.8 (Sample path of a compound Poisson process)** *Simulation of $(X(t_1), \ldots, X(t_n))$ for fixed times $t_1, \ldots, t_n$): a discretized trajectory of the compound Poisson process with parameter $\lambda$.*

- *Simulate a random variable $N$ from Poisson distribution with parameter $\lambda T$ ($N$ gives the total number of jumps on the interval $[0, T]$).*
- *Simulate $N$ i.i.d. random variables, $U_i$, uniformly distributed on the interval $[0, T]$ (These variables correspond to the jump times).*
- *Simulate jump sizes: $N$ independent r.v. $Y_i$ with law $F$.*

*The discrete trajectory of the process is given by*

$$X_t = \sum_{i=1}^{N} \mathbf{1}_{U_i < t} Y_i$$

Sample paths of the various Poisson processes can be seen in the last chapter.

### Jump Diffusion Processes

A Jump-Diffusion process is nothing more than a Brownian motion with drift with in addition a jump part given by a compound Poisson process. We first give an algorithm to simulate a general Jump-Diffusion process with distribution of jumps $F$. Then, we will talk about the two most important Jump-Diffusion models: Merton Normal Jump-Diffusion and Kou Double-Exponential Jump-Diffusion. These two models differ by the distribution of the increments of the jump part. Merton assumes jump size is Gaussian while Kou proposes a Double-Exponential distribution for the jump size.

**Algorithm 4.9 (Sample path of a Jump-Diffusion process)**

*Simulation of* $(X(t_1), \ldots, X(t_n))$ *for* $n$ *fixed times* $t_1, \ldots, t_n$*):*
*Simulate* $n$ *i.i.d.* $N(0,1)$ *random variables*
*Generate* $N \sim Poisson(\lambda(t_{i+1} - t_i))$
*IF* $N_i = 0$
$M_i = 0$
*ELSE Generate* $Y_i$ *from its distribution*
*Set* $M_i = Y_i$

*The discretized trajectory of the process is given by*

$$X_t = \gamma t + \sigma W_t + M$$

In the Merton Normal Jump-Diffusion case, the $Y_i$ have a Normal distribution $(a, b)$. We can thus write $M$ as

$$M = aN + b\sqrt{N}Z$$

where $Z$ is a $N(0,1)$ Normal variable.
In the Kou Double Exponential Jump Diffusion process, $Y_i$ has a Double Exponential distribution. We can see it like that: in the Kou model, $|Y_i|$ has a

Gamma distribution (in fact Exponential) and the sign of $Y_i$ is positive with probability $p$, negative with probability $1-p$. In this case, conditional on the Poisson random variable $N$ taking the value $n$, the number of $Y_i$ with positive sign has a Binomial distribution with parameters $n$ and $p$. We can thus rewrite $M$ as

$$\begin{aligned} K &\sim Binomial(N,p) \\ R_1 &\sim Gamma(K\eta_1,\eta_2) \\ R_2 &\sim Gamma((N-K)\eta_1,\eta_2) \\ M &= R_1 - R_2 \end{aligned}$$

### 4.3.2 Normal Inverse Gaussian Process

First of all, we need an algorithm to generate Inverse Gaussian random variates. Then, we will be able to construct a path of a Normal Inverse Gaussian process by Brownian subordination.

#### Inverse Gaussian Process

**Algorithm 4.10 (IG generator of Michael, Schucany and Haas)**

*Generate a normal random variate* $N$.
*Set* $Y \leftarrow N^2$
*Set* $X_1 \leftarrow (a/b) + (Y/2b^2) - \sqrt{4abY + Y^2}/(2b^2)$
*Generate a uniform* $[0,1]$ *random variate* $U$.
*IF* $U \leq a/(a + X_1 b)$
    *THEN RETURN* $X \leftarrow X_1$
    *ELSE RETURN* $X \leftarrow a^2/(b^2 X_1)$

| Empirical mean | Theoretical mean |
|---|---|
| +2.9979 | +3.0000 |
| Empirical variance | Theoretical variance |
| +0.7592 | +0.7500 |

Table 4.3: IG(6,2) with generator of Michael, Schucany and Haas

The IG random variates generator we have proposed seems to give a great approximation and we can use it without any problem.
We can now construct an IG process.

**Algorithm 4.11 (Sample path of an IG process)** *Simulation of the process $(X(t_1), \ldots, X(t_n))$ for fixed times $t_1, \ldots, t_n$: a discretized trajectory of the IG process with parameters $a, b$.*

- *Simulate $n$ i.i.d. IG random variables $I_t$ with parameter $(ah, b)$ ($h$ is the discretization step)*

*The discrete trajectory of the process is given by*

$$X_{t_i} = X_{t_{i-1}} + I_{t_i}$$

### Normal Inverse Gaussian Process

We can construct a NIG process by Brownian subordination, i.e. we construct an IG process and set is as time parameter for the Brownian motion. The algorithm to construct the simulation is the following.

**Algorithm 4.12 (NIG as a subordinated Brownian Motion)** *Simulation of $(X(t_1), \ldots, X(t_n))$ for fixed times $t_1, \ldots, t_n$: a discretized trajectory of the NIG process with parameters $\alpha, \beta, \delta$.*

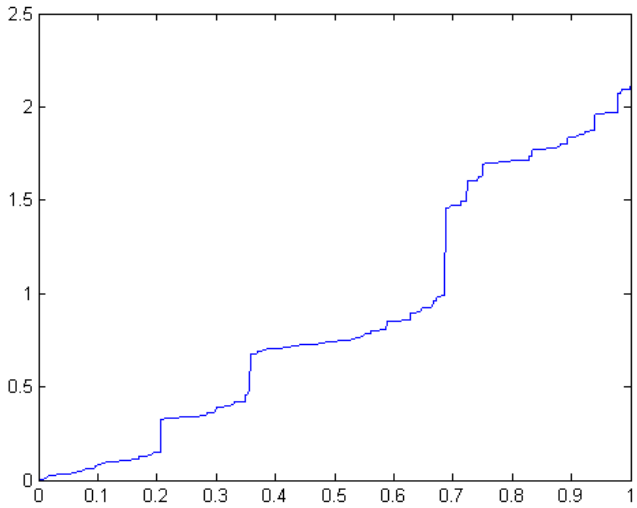



Figure 4.2: An Inverse Gaussian process with parameters $a = 6$ and $b = 2$

*Set* $a = 1$, $b = \delta\sqrt{\alpha^2 - \beta^2}$
*Simulate* $n$ *IG variables* $IG_t$ *with parameters (*$ah$, $b$*) (*$h$ *is the discretization step)*
*Simulate* $n$ *i.i.d.* $N(0, 1)$ *random variables*
*Simulate the process as*
$X_t^{(NIG)} = \beta\delta^2 IG_t + \delta W_{IG_t}$
*where* $W$ *is the Brownian Motion*

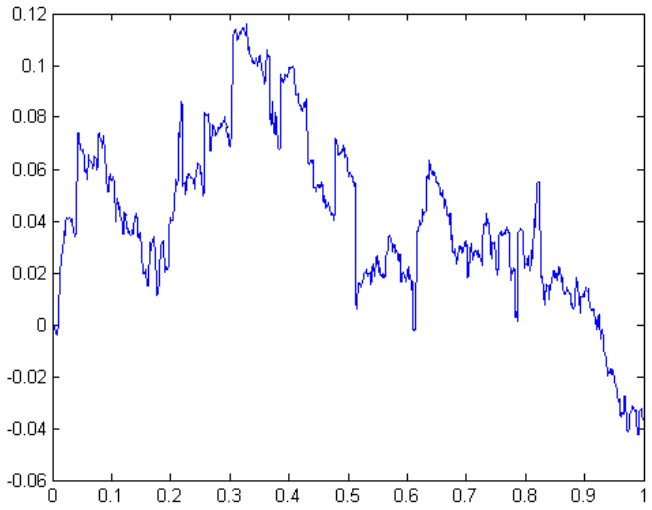



Figure 4.3: A Normal Inverse Gaussian process with parameters $\alpha = 40$, $\beta = -8$ and $\delta = 1$

### 4.3.3 Variance Gamma Process

#### Gamma Process

To simulate a Gamma process, we first need to simulate Gamma numbers from a Gamma$(a, b)$ distribution. First of all, as we saw in the last section

$$cX \sim Gamma(a, \frac{b}{c})$$

Therefore, we just need an algorithm to simulate Gamma$(a, 1)$ numbers. We can then obtain Gamma$(a, b)$ by dividing Gamma$(a, 1)$ numbers by $b$. There are no algorithms to simulate Gamma random numbers which are uniformly fast for all $a$. For most algorithm, we have uniform speed on some interval $[a^*, \infty)$ where $a^*$ is near to 1. For small values of $a$ (i.e. $a < 1$), the algorithm are often not valid, due to the fact that the Gamma density has an infinite

peak at 0 when $a < 1$. So, we cannot use the same method to simulate Gamma numbers with $a \leq 1$ and $a \geq 1$. We need two algorithms.
We first discuss algorithms for $a \geq 1$.
A famous algorithm to generate Gamma variables when $a \geq 1$ is the Best's generator.

**Algorithm 4.13 (Best's generator of Gamma variables)**

*Set* $b \leftarrow a$, $d \leftarrow 3a - 3/4$
*REPEAT*
*Generate i.i.d. uniform* $[0, 1]$ *random variables* $U$, $V$
*Set* $W \leftarrow U(1 - U)$
*Set* $Y \leftarrow \sqrt{\frac{d}{W}}(U - \frac{1}{2})$
*Set* $X \leftarrow c + Y$
*IF* $X < 0$ *go to REPEAT*
*Set* $Z \leftarrow 64W^3V^3$
*UNTIL* $\log(Z) \leq 2(c \log(\frac{X}{c}) - Y)$
*RETURN* $X$

A most recent algorithm to generate Gamma random variates when $a \geq 1$ is the Fishman's generator. It is based upon Cheng and Feast's generator.

**Algorithm 4.14 (Fishman's Algorithm GKM1 - Cheng and Feast)**

*Set* $c \leftarrow 1 - a$, $d \leftarrow (a - (1/6a)))/c$, $m \leftarrow 2/c$, $f \leftarrow m + 2$
*REPEAT*
*Generate i.i.d. uniform* $[0, 1]$ *random variables* $U_1$, $U_2$
*Set* $V \leftarrow (dU_2)/U_1$
*IF* $mU_1 - f + V + (\frac{1}{V}) \leq 0$, *accept*

| Empirical mean | Theoretical mean |
|---|---|
| +1.5224 | +1.5000 |
| +1.5074 | +1.5000 |
| Empirical variance | Theoretical variance |
| +0.9047 | +0.7500 |
| +0.7591 | +0.7500 |

Table 4.4: Gamma(3,2), First line: Best's generator, second line: Fishman's algorithm.

*ELSEIF* $m\log(U_1) - \log(V) + V - 1 \leq 0$, *accept*
*UNTIL accept*
*RETURN* $Z \leftarrow cV$

Best's generator, even though gives a good approximation of the mean seems to be quite bad to approximate the real variance. Therefore, we choose to use Fishman's generator which fits perfectly to the real data.

In the case where $a \leq 1$, we have chosen to present 3 generators:

**Algorithm 4.15 (Johnk's generator of Gamma variables)**

*REPEAT*
*Generate i.i.d. uniform* $[0,1]$ *random variables* $U$, $V$
*Set* $X \leftarrow U^{1/a}$, $Y \leftarrow V^{1/(1-a)}$
*UNTIL* $X + Y \leq 1$
*Generate an exponential random variable* $E$
*RETURN* $(XE)/(X+Y)$

**Algorithm 4.16 (Berman's generator of Gamma variables)**

*REPEAT*

*Generate i.i.d. uniform* $[0,1]$ *random variables* $U_1$, $V_1$

*Set* $X \leftarrow U_1^{1/a}$, $Y \leftarrow V_1^{1/(1-a)}$

*UNTIL* $X + Y \leq 1$

*Generate i.i.d. uniform* $[0,1]$ *random variables* $U_2$, $V_2$

*RETURN* $y \leftarrow -X \log(U_2 V_2)$

**Algorithm 4.17 (Ahrens-Dieter's Gamma generator)**

*Set* $e \leftarrow exp(1)$

*Set* $m \leftarrow (a + e)/e$

*REPEAT*

*Generate i.i.d. uniform* $[0,1]$ *random variables* $U_1$, $U_2$

*Set* $Y \leftarrow mU_1$

*If* $Y \leq 1$

$Z \leftarrow Y^{1/a}$

*If* $U_2 < \exp(-Z)$, *accept*

*Else* $Z \leftarrow -\log(\frac{m-Y}{a})$

*If* $U_2 \leq Z^{a-1}$, *accept*

*UNTIL accept*

*RETURN* $Z$

Johnk and Berman's algorithm seem to give very bad generation of Gamma random variates; both the mean and variance are far from the real data. Therefore, we choose the third algorithm, Ahrens-Dieter's Gamma generator, which fits very well the real data.

We can now generate a sample path of a Gamma process.

| Empirical mean | Theoretical mean |
|---|---|
| +0.2318 | +0.1500 |
| +0.5869 | +0.1500 |
| +0.1495 | +0.1500 |
| Empirical variance | Theoretical variance |
| +0.0817 | +0.0750 |
| +0.4732 | +0.0750 |
| +0.0746 | +0.0750 |

Table 4.5: Gamma(0.3,2), First line: Johnk's generator, second line: Berman's generator, third line: Ahrens-Dieter's generator.

**Algorithm 4.18 (Sample path of a Gamma process)** *Simulation of the process* $(X(t_1), \ldots, X(t_n))$ *for fixed times* $t_1, \ldots, t_n$*): a discretized trajectory of the VG process with parameters* $a, b$*.*

- *Simulate* $n$ *i.i.d. Gamma random variables* $G_t$ *with parameter* $(ah, b)$ *(*$h$ *is the discretization step)*

*The discrete trajectory of the process is given by*

$$X_{t_i} = X_{t_{i-1}} + G_{t_i}$$

### Variance Gamma Process

There are two methods to simulate a Variance Gamma: by Brownian subordination (as a Time-Changed Brownian Motion) or as the difference of two Gamma processes. Below, we present to algorithm to simulate a VG process with each one of these two methods.

**Algorithm 4.19 (Variance Gamma as a subordinated Brownian Motion)** *Simulation of* $(X(t_1), \ldots, X(t_n))$ *for fixed times* $t_1, \ldots, t_n$*: a discretized trajectory of the VG process with parameters* $\sigma, \nu, \theta$*.*

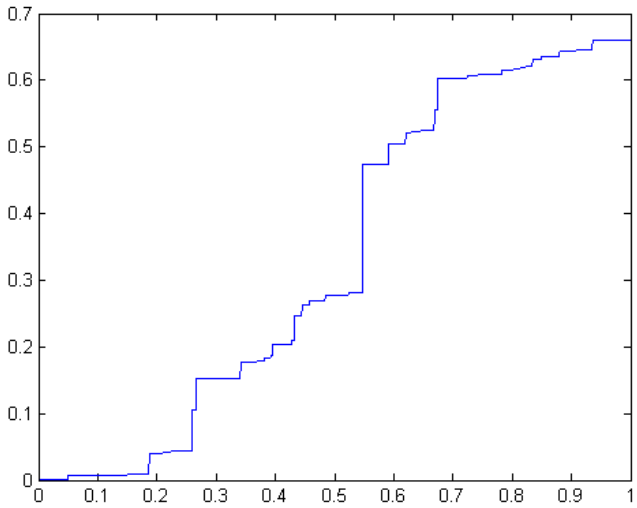



Figure 4.4: A Gamma process with parameters $a = 10$ and $b = 25$

*Set* $a = 1/\nu$, $b = 1/\nu$
*Simulate* $n$ *Gamma variables* $G_t$ *with parameters* $(ah, b)$ *(*$h$ *is the discretization step)*
*Simulate* $n$ *i.i.d.* $N(0,1)$ *random variables*
*Simulate the process as*
$X_t^{(VG)} = \theta G_t + \sigma W_{G_t}$
*where* $W$ *is the Brownian Motion*

**Algorithm 4.20 (Variance Gamma as the difference of two Gamma processes)**
*Simulation of* $(X(t_1), \ldots, X(t_n))$ *for fixed times* $t_1, \ldots, t_n$*: a discretized trajectory of the VG process with parameters* $C, G, M$*.*
*Set* $a_1 = C$, $b_1 = M$
*Set* $a_2 = C$, $b_2 = G$
*Simulate* $n$ *Gamma variables* $G_t^{(1)}$ *with parameters* $(a_1 h,\ b_1)$ *(*$h$ *is the dis-*

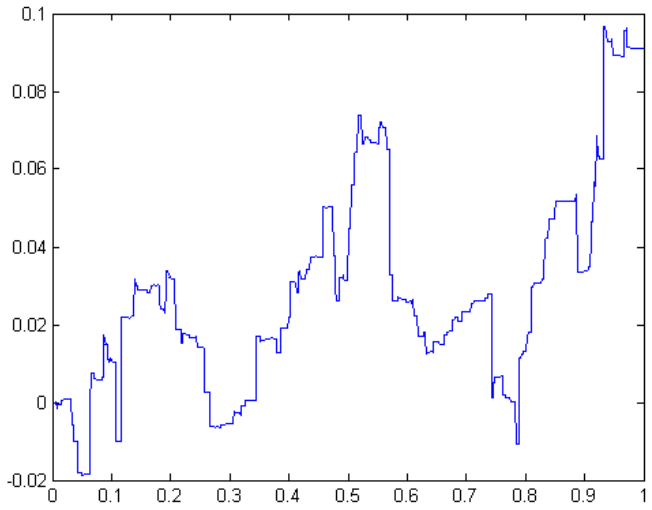



Figure 4.5: A Variance Gamma process with parameters $C = 20$, $G = 80$ and $M = 50$

*cretization step)*
*Simulate n Gamma variables* $G_t^{(2)}$ *with parameters* $(a_2 h,\ b_2)$ *(h is the discretization step)*
*Simulate the process as*
$X_t^{(VG)} = G_t^{(1)} - G_t^{(2)}$

## 4.4 Risk-Neutral Characteristic Functions

Given the result from section 3.3.2, we can give the characteristic function under the risk-neutral equivalent measure for all the Lévy processes we have been studying until now. For simplicity of calculation, we set $S_0 = 1$. In that case, $\ln(S_0) = 0$ and vanishes from the expression of the characteristic

function.

### 4.4.1 Merton Jump-Diffusion Model

The characteristic function of the Merton jump-diffusion model under the measure $\mathbb{P}$ is given by:

$$\Phi(u) = \exp\left\{t\left(i\gamma u - \frac{1}{2}\sigma^2 u^2 + \lambda\left(e^{i\alpha u - \frac{1}{2}\delta^2 u^2} - 1\right)\right)\right\} \tag{4.43}$$

We can then split the characteristic exponent into two parts, a drift part $\mu(u) = i\gamma ut$ and a no-drift part $\varphi(u) = -\frac{1}{2}\sigma^2 u^2 t + \lambda\left(e^{i\alpha u - \frac{1}{2}\delta^2 u^2} - 1\right)t$.
Under the risk-neutral measure, the drift part can be written as

$$\mu^{RN}(u) = i\left(r - \frac{\varphi(-i)}{t}\right)ut = i\left(r - \frac{1}{2}\sigma^2 - \lambda\left(e^{\alpha + \frac{1}{2}\delta^2} - 1\right)\right)ut \tag{4.44}$$

So, the risk-neutral characteristic function of the Merton jump-diffusion model is given by

$$\Phi_{RN}(u) = \exp\left\{t\left[i\Delta u - \frac{1}{2}\sigma^2 u^2 + \lambda\left(e^{i\alpha u - \frac{1}{2}\delta^2 u^2} - 1\right)\right]\right\} \tag{4.45}$$

where $\Delta$ is the risk-neutral drift

$$\Delta = r - \frac{1}{2}\sigma^2 - \lambda(e^{\alpha + \frac{1}{2}\delta^2} - 1) \tag{4.46}$$

And the risk-neutral process for stock price is thus given by

$$\begin{aligned} S_t &= S_0 \exp\left\{X_t^{JD}(\Delta, \sigma, \lambda, \alpha, \delta)\right\} \\ &= \exp\left\{X_t^{JD}(\Delta, \sigma, \lambda, \alpha, \delta)\right\} \qquad as\ S_0 = 1 \end{aligned}$$

where $X_t^{JD}$ is the Merton Jump-Diffusion process and $\Delta(r)$ is the risk-neutral drift.

### 4.4.2 Kou Jump-Diffusion Model

Analogously for the Double Exponential Kou jump-diffusion model:

$$\Phi(u) = \exp\left\{t\left(i\gamma u - \frac{1}{2}\sigma^2 u^2 + \lambda\left(\frac{p\eta_1}{\eta_1 + iu} + \frac{(1-p)\eta_2}{\eta_2 + iu} - 1\right)\right)\right\} \tag{4.47}$$

We can then split the characteristic exponent into two parts, a drift part $\mu(u) = i\gamma ut$ and a no-drift part $\varphi(u) = -\frac{1}{2}\sigma^2 u^2 t + \lambda\left(\frac{p\eta_1}{\eta_1+iu} + \frac{(1-p)\eta_2}{\eta_2+iu} - 1\right)t$. Under the risk-neutral measure, the drift part can be written as

$$\begin{aligned} \mu^{RN}(u) &= i\left(r - \frac{\varphi(-i)}{t}\right)ut \\ &= i\left(r - \frac{1}{2}\sigma^2 - \lambda\left(\frac{p\eta_1}{\eta_1 + 1} + \frac{(1-p)\eta_2}{\eta_2 + 1} - 1\right)\right)ut \end{aligned} \tag{4.48}$$

So, the risk-neutral characteristic function of the Double Exponential Kou jump-diffusion model is given by

$$\Phi_{RN}(u) = \exp\left\{t\left[i\Delta u - \frac{1}{2}\sigma^2 u^2 + \lambda\left(\frac{p\eta_1}{\eta_1 + iu} + \frac{(1-p)\eta_2}{\eta_2 + iu} - 1\right)\right]\right\} \tag{4.49}$$

where $\Delta$ is the risk-neutral drift

$$\Delta = r - \frac{1}{2}\sigma^2 - \lambda\left(\frac{p\eta_1}{\eta_1 + 1} + \frac{(1-p)\eta_2}{\eta_2 + 1} - 1\right) \tag{4.50}$$

And the risk-neutral process for stock price is thus given by

$$\begin{aligned} S_t &= S_0 \exp\left\{X_t^{DE}(\Delta, \sigma, \lambda, p, \eta_1, \eta_2)\right\} \\ &= \exp\left\{X_t^{DE}(\Delta, \sigma, \lambda, p, \eta_1, \eta_2)\right\} \qquad as\ S_0 = 1 \end{aligned}$$

where $X_t^{DE}$ is the Kou Jump-Diffusion process and $\Delta(r)$ is the risk-neutral drift.

### 4.4.3 Normal Inverse Gaussian Model

Analogously for the Normal Inverse Gaussian:

$$\Phi(u) = \exp\left\{-\delta t\left(\sqrt{\alpha^2-(\beta+iu)^2}-\sqrt{\alpha^2-\beta^2}\right)\right\} \tag{4.51}$$

We can then split the characteristic exponent into two parts, a drift part $\mu(u) = 0$ and a no-drift part $\varphi(u) = -\delta t\left(\sqrt{\alpha^2-(\beta+iu)^2}-\sqrt{\alpha^2-\beta^2}\right)$. Under the risk-neutral measure, the drift part can be written as

$$\mu^{RN}(u) = i\left(r-\frac{\varphi(-i)}{t}\right)ut = i\left(r+\delta\left(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\right)\right)ut \tag{4.52}$$

So, the risk-neutral characteristic function of the Normal Inverse Gaussian model is given by

$$\Phi_{RN}(u) = \exp\left\{i\Delta u-\delta t\left(\sqrt{\alpha^2-(\beta+iu)^2}-\sqrt{\alpha^2-\beta^2}\right)\right\} \tag{4.53}$$

where $\Delta$ is the risk-neutral drift

$$\Delta = r+\delta\left(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\right) \tag{4.54}$$

And the risk-neutral process for stock price is thus given by

$$\begin{aligned} S_t &= S_0\exp\left\{\Delta t+X_t^{NIG}(\delta,\alpha,\beta)\right\} \\ &= \exp\left\{\Delta t+X_t^{NIG}(\delta,\alpha,\beta)\right\} \qquad as\ S_0=1 \end{aligned}$$

where $X_t^{NIG}$ is the NIG process and $\Delta(r)$ is the risk-neutral drift.

### 4.4.4 Variance Gamma Model

As seen before, we can formulate the characteristic function of the Variance Gamma model under the measure $\mathbb{P}$ as

$$\Phi(u) = \left(1-iu\theta\nu+\frac{1}{2}\sigma^2u^2\nu\right)^{-t/\nu} \tag{4.55}$$

or

$$\Phi(u) = \left(\frac{GM}{GM + (M-G)iu + u^2}\right)^{Ct} \tag{4.56}$$

First, we present the risk-neutral form for the first formulation and next for the second. The first characteristic function under the measure $\mathbb{P}$ can be rewritten as

$$\begin{aligned}\Phi(u) &= \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2u^2\nu\right)^{-t/\nu} \\ &= \exp\left(\ln\left(1 - iu\theta\nu + \frac{1}{2}\sigma^2u^2\nu\right)^{-t/\nu}\right) \\ &= \exp\left(-\frac{t}{\nu}\ln\left(1 - iu\theta\nu + \frac{1}{2}\sigma^2u^2\nu\right)\right)\end{aligned} \tag{4.57}$$

We can then split the characteristic exponent into two parts, a drift part $\mu(u) = 0$ and a no-drift part $\varphi(u) = -\frac{t}{\nu}\ln\left(1 - iu\theta\nu + \frac{1}{2}\sigma^2u^2\nu\right)$.
Under the risk-neutral measure, the drift part can be written as

$$\mu^{RN}(u) = i\left(r - \frac{\varphi(-i)}{t}\right)ut = i\left(r + \frac{\ln\left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right)}{\nu}\right)ut \tag{4.58}$$

So, the risk-neutral characteristic function of the Variance Gamma model (1) is given by

$$\Phi_{RN}(u) = \exp\left\{i\Delta ut - \frac{t}{\nu}\ln\left(1 - iu\theta\nu + \frac{1}{2}\sigma^2u^2\nu\right)\right\} \tag{4.59}$$

or

$$\Phi_{RN}(u) = e^{i\Delta ut}\left(1 - iu\theta\nu + \frac{1}{2}\sigma^2u^2\nu\right)^{-t/\nu} \tag{4.60}$$

where $\Delta$ is the risk-neutral drift

$$\Delta = r + \frac{\ln\left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right)}{\nu} \tag{4.61}$$

And the risk-neutral process for stock price is thus given by

$$\begin{aligned} S_t &= S_0 \exp\left\{\Delta t + X_t^{VG}(\sigma,\theta,\nu)\right\} \\ &= \exp\left\{\Delta t + X_t^{VG}(\sigma,\theta,\nu)\right\} \qquad as\ S_0 = 1 \end{aligned}$$

where $X_t^{VG}$ is the VG process and $\Delta(r)$ is the risk-neutral drift.

On the other hand, the second characteristic function under the measure $\mathbb{P}$ can be rewritten as

$$\begin{aligned} \Phi(u) &= \left(\frac{GM}{GM+(M-G)iu+u^2}\right)^{Ct} \\ &= \exp\left(\ln\left[\left(\frac{GM}{GM+(M-G)iu+u^2}\right)^{Ct}\right]\right) \\ &= \exp\left(Ct\ln\left(\frac{GM}{GM+(M-G)iu+u^2}\right)\right) \end{aligned} \tag{4.62}$$

We can then split the characteristic exponent into two parts, a drift part $\mu(u) = 0$ and a no-drift part $\varphi(u) = Ct\ln\left(\frac{GM}{GM+(M-G)iu+u^2}\right)$.
Under the risk-neutral measure, the drift part can be written as

$$\mu^{RN}(u) = i\left(r - \frac{\varphi(-i)}{t}\right)ut = i\left(r - C\ln\left(\frac{GM}{GM+(M-G)-1}\right)\right)ut \tag{4.63}$$

As $\ln\left(\frac{A}{B}\right) = -\ln\left(\frac{B}{A}\right)$ and $GM+(M-G)-1 = (M-1)(G+1)$, we can rewrite $\mu^{RN}(u)$ as

$$\mu^{RN}(u) = i\left(r + C\ln\left(\frac{(M-1)(G+1)}{GM}\right)\right)ut \tag{4.64}$$

So, the risk-neutral characteristic function of the Variance Gamma model (2) is given by

$$\Phi_{RN}(u) = \exp\left\{i\Delta ut + Ct\ln\left(\frac{GM}{GM+(M-G)iu+u^2}\right)\right\} \tag{4.65}$$

| Model | $\varphi(u)$ |
|---|---|
| BS($\sigma > 0$) | $-\frac{1}{2}\sigma^2 u^2 t$ |
| JD($\sigma > 0, \lambda > 0, \delta > 0$) | $-\frac{1}{2}\sigma^2 u^2 t + \lambda t \left(e^{iu\alpha - \frac{1}{2}u^2\delta^2} - 1\right)$ |
| DE($\sigma > 0, \lambda > 0, p > 0, \eta_1 > 0, \eta_2 > 0$) | $-\frac{1}{2}\sigma^2 u^2 t + \lambda t \left(\frac{p\eta_1}{\eta_1 + iu} + \frac{(1-p)\eta_2}{\eta_2 + iu} - 1\right)$ |
| NIG($\alpha > 0, -\alpha < \beta < \alpha, \delta > 0$) | $-\delta t \left(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}\right)$ |
| VG1($\sigma > 0, \nu > 0$) | $-\frac{t}{\nu}\ln\left(1 - iu\theta\nu + \frac{1}{2}\sigma^2 u^2 \nu\right)$ |
| VG2($C > 0, G > 0, M > 0$) | $Ct \ln\left(\frac{GM}{GM + (M-G)iu + u^2}\right)$ |

Table 4.6: No-drift part of the characteristic exponent of some parametric Lévy processes (BS: Black-Scholes, JD: Merton jump-diffusion, DE: Kou double-exponential jump-diffusion, NIG: Normal Inverse Gaussian, VG: Variance Gamma)

or

$$\Phi_{RN}(u) = e^{i\Delta ut}\left(\frac{GM}{GM + (M-G)iu + u^2}\right)^{Ct} \tag{4.66}$$

where $\Delta$ is the risk-neutral drift

$$\Delta = r + C\ln\left(\frac{(M-1)(G+1)}{GM}\right) \tag{4.67}$$

As a summary, we have presented the drift and no-drift parts of the characteristic function of the various models in the tables 4.6 and 4.7. We remember that the risk-neutral characteristic function is defined (when $S_0 = 1$) as

$$\Phi_{RN}(u) = e^{\varphi(u) + \mu^{RN}(u)} = e^{\varphi(u) + i\Delta ut} \tag{4.68}$$

| Model | $\Delta = r - \frac{\varphi(-i)}{t}$ |
|---|---|
| BS | $r - \frac{1}{2}\sigma^2$ |
| JD | $r - \frac{1}{2}\sigma^2 - \lambda\left(e^{\alpha+\frac{1}{2}\delta^2} - 1\right)$ |
| DE | $r - \frac{1}{2}\sigma^2 - \lambda\left(\frac{p\eta_1}{\eta_1+1} + \frac{(1-p)\eta_2}{\eta_2+1} - 1\right)$ |
| NIG | $r + \delta\left(\sqrt{\alpha^2-(\beta+1)^2} - \sqrt{\alpha^2-\beta^2}\right)$ |
| VG1 | $r + \frac{\ln\left(1-\theta\nu-\frac{1}{2}\sigma^2\nu\right)}{\nu}$ |
| VG2 | $r + C\ln\left(\frac{(M-1)(G+1)}{MG}\right)$ |

Table 4.7: Drift parameter under the risk-neutral measure

# Chapter 5

# Option Pricing with FRFT

Several methods can be used to price an option under a Lévy process. One can derive the partial differential equation PDE (in particular partial integro-differential equation PIDE). Then, a lot of numerical methods, like finite differences for example, can be used to solve (numerically) these PDEs. However this method is a bit complex given the jump part.

Another method, easy to implement and also available for exotic products is the Monte-Carlo method. In this setting, we just need to simulate an important number of paths of the risk neutral process, take the average final value and take its discounted value to recover the option price. However, this method is computationally heavy because one needs to simulate processes with a little discretization step to obtain good results and then take a lot of paths to ensure a good approximation. However, there exist some tricks to simulate a jump-diffusion process on a few points and then to be able to price exotic options in an easy way.

A third method is based on the fact that the probability density of a Lévy process is often not known in closed form but its characteristic function al-

ways is. Due to this fact, Fourier-based option pricing methods have been developed for exponential-Lévy models. These methods require the use of te Fourier transform. They are very well performing since the Fourier transform can be efficiently computed using fast algorithm, so that the overall complexity of the algorithm for option price is comparable with the cost for evaluating the Black-Scholes formula. Below, we develop the analytic form of the Fourier transform of the option price and then explain the FFT and FRFT algorithms to calculate the anti-transform and to recover the option price.

## 5.1 An Analytic Expression for the Fourier Transform

In this section, we follow Carr and Madan's 1999 paper [24] which gives a revolutionary method based on Fourier transform and the use of the FFT algorithm for option pricing when the characteristic function of the log-price is known analytically. They develop an analytic expression for the Fourier transform of an option price (or its time value, see below). Then, they recover the option price by making the anti-transform with FFT. In this thesis, we adapt the methodology of Carr and Madan [24] in order to use the fractional FFT (FRFT), developed by Bailey and Swarztrauber in 1991 and 1994 [8] [9] and recently used by Chourdakis [31] for option pricing.

We begin explaining why we work with log-price $s_T = \ln(S_T)$. Since we choose

$S_0 = 1$, we have

$$\begin{aligned} S_T &= \exp(L_T) \\ s_T &= \ln(\exp(L_T)) = L_T \end{aligned} \tag{5.1}$$

The log-price process is then totally described by the risk-neutral Lévy process we use. The characteristic function of the log-price is thus the risk-neutral characteristic function of the Lévy process, that is

$$\phi_T(u) = \mathbf{E}[\exp(ius_T)] = \mathbf{E}[\exp(iuL_T)] \tag{5.2}$$

Moreover, even though any risk-neutral Lévy process can take negative values, the exponential of this process is always positive, which is a requisite for a process describing the path of a stock price.

### 5.1.1 Modified Option Price

Let be $s_T := \ln(S_T)$ and $k := \ln(K)$, where $K$ is the strike price of the option. Then the value of a European call option with maturity $T$ as a function of $k$ is given by

$$C_T(k) = \int_k^\infty e^{-rT} \left(e^s - e^k\right) q_T(s) ds \tag{5.3}$$

where $q_T(s)$ is the risk-neutral density function of $s$.
Since $C_T(k)$ is not square integrable ($C_T(k)$ tends to $S_0$ as $k$ tends to $-\infty$) Carr and Madan define a modified call price function:

$$c_T(k) = e^{\alpha k} C_T(k), \qquad \alpha > 0. \tag{5.4}$$

We now expect $c_T(k)$ to be square integrable for a range of $\alpha$ values and $\forall k$. The parameter $\alpha$ is referred to as the *dampening factor*. The Fourier transform and inverse Fourier transform of $c_T(k)$ are given by

$$\mathbf{F}_{c_T}(v) = \psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} c_T(k) dk \tag{5.5}$$

$$c_T(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \mathbf{F}_{c_T}(v) dv = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_T(v) \tag{5.6}$$

Thus

$$\begin{aligned} C_T(k) &= e^{-\alpha k} c_T(k) \\ &= e^{-\alpha k} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_T(v) \end{aligned} \tag{5.7}$$

Since

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{-ivk} \psi_T(v) &= \int_{0}^{+\infty} e^{-ivk} \psi_T(v) + \int_{-\infty}^{0} e^{-ivk} \psi_T(v) \\ &= \int_{0}^{+\infty} e^{-ivk} \psi_T(v) + \int_{0}^{+\infty} e^{-ivk} \psi_T(-v) \end{aligned} \tag{5.8}$$

Note that when $\psi_T(v)$ is symmetric we have $\psi_T(v) = \psi_T(-v)$ then

$$\int_{-\infty}^{+\infty} e^{-ivk} \psi_T(v) = 2Re \int_{0}^{+\infty} e^{-ivk} \psi_T(v) \tag{5.9}$$

$Re(.)$ stands for the real part. As $C_T(k)$ is real

$$C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-ivk} \psi_T(v) \tag{5.10}$$

We just have to retrieve an analytic form for $\psi_T(v)$ to put in the equation (5.10) to get back the price of the option.

Carr and Madan obtain the following form for $\psi_T(v)$

$$\psi_T(v) = \frac{e^{-rT}\Phi_T(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v} \tag{5.11}$$

We explain how to derive this result.

Let be

$$\begin{aligned}\psi_T(v) &= \int_{-\infty}^{\infty} e^{ivk} c_T(k)dk, \qquad\qquad c_T(k) = e^{\alpha k}C_T(k)\\ C_T(k) &= \int_k^{\infty} e^{-rT}\left(e^s - e^k\right) q_T(s)ds\end{aligned}$$

we can write

$$\begin{aligned}\psi_T(v) &= \int_{-\infty}^{\infty} e^{ivk}\int_k^{\infty} e^{\alpha k}e^{-rT}\left(e^s-e^k\right)q_T(s)dsdk &(5.12)\\ &= \int_{-\infty}^{\infty} e^{-rT}q_T(s)\int_{-\infty}^{s} e^{\alpha k}\left(e^s-e^k\right)e^{ivk}dkds\\ &= \int_{-\infty}^{\infty} e^{-rT}q_T(s)\int_{-\infty}^{s}\left(e^{s+\alpha k}-e^{(1+\alpha)k}\right)e^{ivk}dkds &(5.13)\\ &= \int_{-\infty}^{\infty} e^{-rT}q_T(s)\int_{-\infty}^{s}\left(e^{s+(\alpha+iv)k}-e^{(1+\alpha+iv)k}\right)dkds\\ &= \int_{-\infty}^{\infty} e^{-rT}q_T(s)\left[\int_{-\infty}^{s} e^{s+(\alpha+iv)k}dk - \int_{-\infty}^{s} e^{(1+\alpha+iv)k}dk\right]ds\end{aligned}$$

Since the following equation holds:

$$\int_a^b f'(x)e^{f(x)}dx = e^{f(x)} \tag{5.14}$$

we can rewrite

$$\begin{aligned}\int_{-\infty}^{s} e^{s+(\alpha+iv)k}dk &= \left[\frac{e^{s+(\alpha+iv)k}}{\alpha+iv}\right]_{-\infty}^{s} = \frac{e^{s+(\alpha+iv)s}}{\alpha+iv} - \lim_{k\to-\infty}\left(\frac{e^{s+(\alpha+iv)k}}{\alpha+iv}\right)\\ &= \frac{e^{(\alpha+1+iv)s}}{\alpha+iv}\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{s} e^{(1+\alpha+iv)k}dk &= \left[\frac{e^{(1+\alpha+iv)k}}{\alpha+1+iv}\right]_{-\infty}^{s} = \frac{e^{(1+\alpha+iv)s}}{\alpha+1+iv} - \lim_{k\to-\infty}\left(\frac{e^{(1+\alpha+iv)k}}{\alpha+1+iv}\right) \\ &= \frac{e^{(1+\alpha+iv)s}}{\alpha+1+iv}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\psi_T(v) &= \int_{-\infty}^{\infty} e^{-rT} q_T(s)\left[\frac{e^{(\alpha+1+iv)s}}{\alpha+iv} - \frac{e^{(1+\alpha+iv)s}}{\alpha+1+iv}\right]ds \qquad (5.15)\\ &= \int_{-\infty}^{\infty} e^{-rT} q_T(s)\left[\frac{(\alpha+1+iv)e^{(\alpha+1+iv)s} - (\alpha+iv)e^{(\alpha+1+iv)s}}{(\alpha+iv)(\alpha+1+iv)}\right]ds \\ &= \int_{-\infty}^{\infty} e^{-rT} q_T(s)\left[\frac{e^{(\alpha+1+iv)s}}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}\right]ds \\ &= \frac{e^{-rT}}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}\int_{-\infty}^{\infty} q_T(s)e^{(\alpha+1+iv)s}ds.\end{aligned}$$

It is easy to see that the following relation holds

$$\begin{aligned}(\alpha+1+iv)s &= \left(\frac{\alpha i}{i} + \frac{i}{i} + iv\right)s \\ &= i\left(\frac{\alpha}{i} + \frac{1}{i} + v\right)s \\ &= i\left(-i\alpha - i + v\right)s \\ &= i\left(v - (\alpha+1)i\right)s\end{aligned}$$

so that since we have

$$\Phi_T(u) = \int_{-\infty}^{\infty} q_T(s)e^{ius}ds$$

and

$$\psi_T(v) = \frac{e^{-rT}}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}\int_{-\infty}^{\infty} q_T(s)e^{i(v-(\alpha+1)i)s}ds$$

we obtain the following expression

$$\psi_T(v) = \frac{e^{-rT}\Phi_T(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}. \qquad (5.16)$$

This concludes the proof.

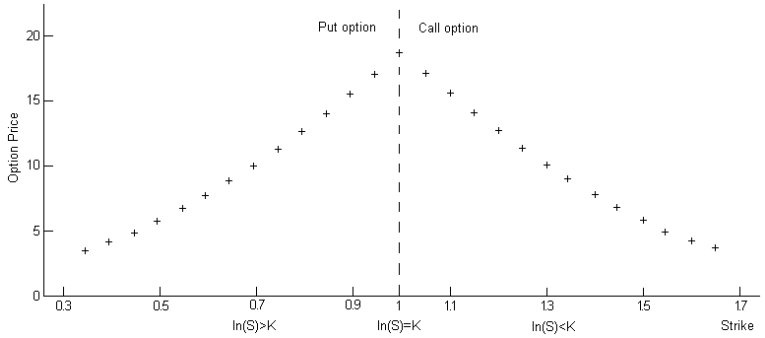



Figure 5.1: The function $z_T(k)$.

### 5.1.2 Time Value of an Option

The formula derived in the previous section depends on the intrinsic value of the option. However, when dealing with out-of-the-money options, there is no intrinsic value. A Fourier transform technique based on the time value of the option has to be derived.

Carr and Madan [24] developed a formula for out-of-the-money options based on the time value of the option. We derive this formula but we don't implement it since the authors find very little difference between the results obtained by using the two formulas. Again, let $s_T := \ln(S_T)$ and $k := \ln(K)$, where $K$ is the strike price of the option and $S_0$ the initial spot price. Let $z_T(k)$ be the $T$ maturity put price when $k < \ln(S_0)$ and the $T$ maturity call price when $k > \ln(S_0)$. As shown in Figure (5.1), the function $z_T(k)$ is peaked as $k = \ln(S_0)$ and declines in both directions as $k$ goes to $-\infty$ and to $+\infty$.

Below, we explain the development of an analytic expression for the Fourier transform of $z_T(k)$ in terms of the characteristic function of $s_T$.

Let $\zeta_T(v)$ denote the Fourier transform of $z_T(k)$

$$\zeta_T(v) = \int_{-\infty}^{+\infty} e^{ivk} z_T(k) dk \tag{5.17}$$

The prices of out-of-the-money options are obtained by inverting this transform

$$z_T(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk} \zeta_T(v) dv \tag{5.18}$$

Let's give an analytic form for $z_T(k)$

$$z_T(k) = e^{-rT} \int_{-\infty}^{+\infty} \left\{ (e^k - e^s) \mathbf{1}_{s<k,k<0} + (e^s - e^k) \mathbf{1}_{s>k,k>0} \right\} q_T(s) ds \tag{5.19}$$

where $q_T$ is the risk-neutral density. Then, we have

$$\begin{aligned}
\zeta_T(v) &= \int_{-\infty}^{+\infty} e^{ivk} e^{-rT} \int_{-\infty}^{+\infty} \left\{ (e^k - e^s) \mathbf{1}_{s<k,k<0} + (e^s - e^k) \mathbf{1}_{s>k,k>0} \right\} q_T(s) ds dk \\
&= \int_{-\infty}^{0} e^{ivk} e^{-rT} dk \int_{-\infty}^{k} (e^k - e^s) q_T(s) ds \\
&\quad + \int_{0}^{+\infty} e^{ivk} e^{-rT} dk \int_{k}^{+\infty} (e^s - e^k) q_T(s) ds \\
&= \int_{-\infty}^{0} e^{-rT} q_T(s) ds \int_{s}^{+\infty} (e^{(1+iv)k} - e^s e^{ivk}) dk \\
&\quad + \int_{0}^{+\infty} e^{-rT} q_T(s) ds \int_{0}^{s} (e^s e^{ivk} - e^{(1+iv)k}) dk
\end{aligned} \tag{5.20}$$

Formula (5.20) after simplification gives

$$\zeta_T(v) = e^{-rT} \left[ \frac{1}{1+iv} - \frac{e^{rt}}{iv} - \frac{\Phi_T(v-i)}{v^2 - iv} \right] \tag{5.21}$$

However, when maturity is small ($T \to 0$), the function $z_T(k)$ approximates the shape of a Dirac delta function and is difficult to invert. Therefore, it is useful to consider the function $\sinh(\alpha k) z_T(k)$ since this function vanished at

$k = 0$. Hence we have

$$\begin{aligned}
\gamma_T(v) &= \int_{-\infty}^{+\infty} e^{ivk} \sinh(\alpha k) z_T(k) dk \qquad (5.22)\\
&= \int_{-\infty}^{+\infty} e^{ivk} \frac{e^{\alpha k} - e^{-\alpha k}}{2} z_T(k) dk \\
&= \int_{-\infty}^{+\infty} \frac{e^{(iv+\alpha)k} - e^{(iv-\alpha)k}}{2} z_T(k) dk \\
&= \frac{1}{2}\left[\int_{-\infty}^{+\infty} e^{(iv+\alpha)k} z_T(k) dk - \int_{-\infty}^{+\infty} e^{(iv-\alpha)k} z_T(k) dk\right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{+\infty} e^{i(v-i\alpha)k} z_T(k) dk - \int_{-\infty}^{+\infty} e^{i(v+i\alpha)k} z_T(k) dk\right] \\
&= \frac{\zeta_T(v - i\alpha) - \zeta_T(v + i\alpha)}{2}.
\end{aligned}$$

The time value is thus given by:

$$\begin{aligned}
z_T(k) &= \frac{1}{\sinh(\alpha k)} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk} \gamma_T(v) dv \qquad (5.23)\\
\gamma_T(v) &= \frac{\zeta_T(v - i\alpha) - \zeta_T(v + i\alpha)}{2} \qquad (5.24)\\
\zeta_T(k) &= e^{-rT}\left[\frac{1}{1+iv} - \frac{e^{rt}}{iv} - \frac{\Phi_T(v-i)}{v^2 - iv}\right] \qquad (5.25)
\end{aligned}$$

## 5.2 FFT

Now we have an analytic expression for the Fourier transform of the option price, we have to look for a method to invert this transform (take the anti-transform) to recover the option price. The well known method to do this is the Fast Fourier Transform (FFT) algorithm. Roughly speaking, the FFT, developed by Cooley and Tukey in 1965 [34], consists in approximating the

continuous Fourier transform (CFT) with its discrete counterpart (DFT)

$$\int_0^\infty e^{-ixu}h(u)du \;\approx\; \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}kj}h_j \qquad\qquad k=0,\ldots,N-1 \tag{5.26}$$

The FFT is thus an easy and fast way to compute sums of type $\sum_{j=0}^{N-1} e^{i\frac{2\pi}{N}kj}h_j$.

**Definition 5.1 (Trapezoidal Rule)**

$$\int_a^b f(x)dx \approx \Delta x\left[\frac{1}{2}f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\frac{1}{2}f(x_n)\right] \tag{5.27}$$

*where*

$$\Delta x = \frac{(b-a)}{n} \qquad \text{and} \qquad x_j = a+j\Delta x$$

Using an integration rule as the trapezoidal rule, we can rewrite the integral that appears in the formula (5.10) for in the option price as follows:

$$\int_0^\infty e^{-ivk}\psi(v)dv \approx \sum_{j=0}^{N-1} e^{-iv_jk}\psi'(v_j)\eta \tag{5.28}$$

where $\eta$ is the discretization step or grid spacing ($\Delta x$ in the definition of the trapezoidal rule), the points $v_j$ are chosen to be equidistant with grid spacing $\eta$, that is $v_j = \eta j$. Finally $\psi'$ is just $\psi$ weighted by the integration rule. The upper limit of the integration is thus $a = \eta N$ so the value of $\eta$ has to be small enough to allow a good approximation but not too small to guarantee that the characteristic function is equal to zero for any point $a' > a$.

We can clearly see that the sum in (5.28) is not a direct application of the FFT algorithm. Thus, we cannot use the FFT algorithm on that summation, we need to modificate it. In general, the strikes near the spot price are of

interest since such options are traded most frequently. We consider log-strikes equally spaced around the log spot price $s_0$.

$$k_u = -\frac{1}{2}N\lambda + \lambda u + s_0, \qquad for \qquad u = 0, \ldots, N-1 \tag{5.29}$$

However, when we set $S_0 = 1$, we obtain $s_0 = 0$ and this implies $k_u = -\frac{1}{2}N\lambda + \lambda u$ where $\lambda$ denotes the distance between two consecutive log-strikes. We take thus log-strikes ranging from $-b$ to $b$ where $b = \frac{1}{2}N\lambda$.

The summation (5.28) can be rewritten as

$$\begin{aligned}
\sum_{j=0}^{N-1} e^{-iv_j k_u}\psi'(v_j)\eta &= \sum_{j=0}^{N-1} e^{-i\left(-\frac{1}{2}N\lambda+\lambda u\right)v_j}\psi'(v_j)\eta \\
&= \sum_{j=0}^{N-1} e^{-i\left(-\frac{1}{2}N\lambda+\lambda u\right)\eta j}\psi'(v_j)\eta \\
&= \sum_{j=0}^{N-1} e^{-i\lambda u\eta j} e^{i\frac{N\lambda}{2}\eta j}\psi'(v_j)\eta
\end{aligned}$$

If we set $h_j = e^{i\frac{N\lambda}{2}\eta j}\psi'(v_j)\eta$ and $\lambda\eta = \frac{2\pi}{N}$ we can rewrite the summation as follows:

$$\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}uj} h_j \tag{5.30}$$

and we can apply the FFT algorithm on the vector $h_j = e^{i\frac{N\lambda}{2}\eta j}\psi'(v_j)\eta$ provided that:

$$\lambda\eta = \frac{2\pi}{N}. \tag{5.31}$$

It is clear that among the three parameters $\eta$, $\lambda$ and $N$ only two can be arbitrarily chosen, the third must satisfy the condition (5.31). Since the parameters $\eta$ and $N$ are chosen to make the integration accurate enough, the value of the strike grid spacing $\lambda$ will be inversely proportional to the upper integration

bound. Since the resulting grid spacing may not be dense enough for option pricing, we will have to increase $N$, actually increasing consistently the computation time. So we face a trade-off between accuracy and the number of strikes.

In fact, the major problem of option pricing with FFT algorithm is the condition (5.31). For option pricing, we need an integration grid very dense to insure a good integral approximation but also a very dense log-strike grid since we need a consequent number of option prices for calibration. As the condition (5.31) in the FFT procedure the integration grid spacing $\eta$ is inversely proportional to the log-strike grid spacing $\lambda$, the unique solution to increase both densities (to reduce both $\eta$ and $\lambda$) is to increase $N$.

We need a more efficient method, which transcends the condition (5.31), to calibrate our model.

## 5.3 FRFT

The Fractional FFT is an easy and fast way to compute sums of type

$$\sum_{j=0}^{N-1} e^{-i2\pi kj\epsilon} h_j \tag{5.32}$$

We clearly denote that FFT is a special case of FRFT where $\epsilon = \frac{1}{N}$. Fractional FFT algorithm needs three FFT procedures, that is to say an FRFT procedure is three time longer than an FFT one taken on the the same input vector. However, the benefits of using FRFT is that both grid $\eta$ and $\lambda$ can be chosen independently. Practically, we can obtain a good integral approximation taking a fine integration grid and taking the log-strikes around ATM strike without having to span a huge vector with zeros, providing a huge computational time.

As input vectors are shorter in the FRFT case, the resulting computational time of an FRFT procedure is much shorter than its FFT counterpart.
Let's show an example. Carr and Madan, in their paper, took a 4096 point FFT with upper integration bound 1024. Since the upper integration is chosen to be $a = \eta N$, the integration grid step is $\eta = 0.25$. Since we have $\lambda\eta = \frac{2\pi}{N}$ then the log-strike grid step is $\lambda = 0.006$. However, only 67 out of the 4096 points of the log-strike grid fall in the interval ATM strike $\pm 20\%$, which is of interest for option pricing. In this way, more than 4000 option prices calculated by using the FFT procedure are not required. Analogously, out of the 4096 points that are input in the FFT procedure (values of the integrand), only 80 are greater than $10^{-8}$ that is to say significantly different from zero.
Following Bailey and Swartztrauber [8] and [9] and Chourdakis [31], let's now take a look on the implementation of the FRFT. Suppose that we want to compute an N-point FRFT on a sequence (or vector) $x$. The algorithm can be stated as follows. Define two 2N-points vectors $y$ and $z$ as

$$y_j = x_j e^{-\pi i j^2 \epsilon} \qquad 0 \leq j < m \tag{5.33}$$

$$y_j = 0 \qquad m \leq j < 2m \tag{5.34}$$

$$z_j = e^{\pi i j^2 \epsilon} \qquad 0 \leq j < m \tag{5.35}$$

$$z_j = e^{\pi i (j-2m)^2 \epsilon} \qquad m \leq j < 2m \tag{5.36}$$

where $\epsilon$ is the fractional parameter given by $\epsilon = \frac{1}{N} = \frac{\eta\lambda}{2\pi}$

The FRFT is given by

$$G_k(x, \epsilon) = e^{-\pi i k^2 \epsilon} \odot D_k^{-1} \left[ D_j(y) \odot D_j(z) \right] \tag{5.37}$$

where $\odot$ means element-wise multiplication and $D_j(.)$ is the DFT $\sum_{j=0}^{N-1} e^{i\frac{2\pi}{N}k_j} h_j$ calculated with the FFT algorithm.

To compute the FRFT algorithm, one just needs to initially choose the parameters, that is the number of points $N$, the upper integration bound $a$ and the bound for log prices $b$ (prices ranged on $[e^{-b}, e^{b}]$). The integration grid spacing is then given by $\eta = a/N$, the log-strikes grid spacing by $\lambda = 2b/N$ and the fractional parameter by $\epsilon = \frac{\eta\lambda}{2\pi}$. Then, the two grids can be described as it follows. The input grid is spanned on the interval $(a, N)$ with grid spacing $\eta$ and the output grid on the interval $(-b, b)$ with grid spacing $\lambda$. As in the case of the FFT, we evaluate the function $\psi$ on the input grid and apply the integration rule (trapezoidal rule). Then we apply the FRFT routine, recover the normalized prices and storage two vectors, one with the strike prices and one with the option prices.

## 5.4 Results

### 5.4.1 The Dampening Factor $\alpha$

As seen previously, since $C_T(k)$ is not square integrable we have to use a modified option price which directly depends on a dampening factor $\alpha$. As this factor cannot be chosen randomly, in this section, we analyze the impact of the dampening factor $\alpha$ on the option price.

We calculate various option prices with the FRFT algorithm taking the same parameters and choosing the value of the dampening parameters ranging in the interval $[0, 7]$. From the Figure (5.2) we see that for values of $\alpha$ greater than 0.5 (in this case, for the given parameters), the option price given by the

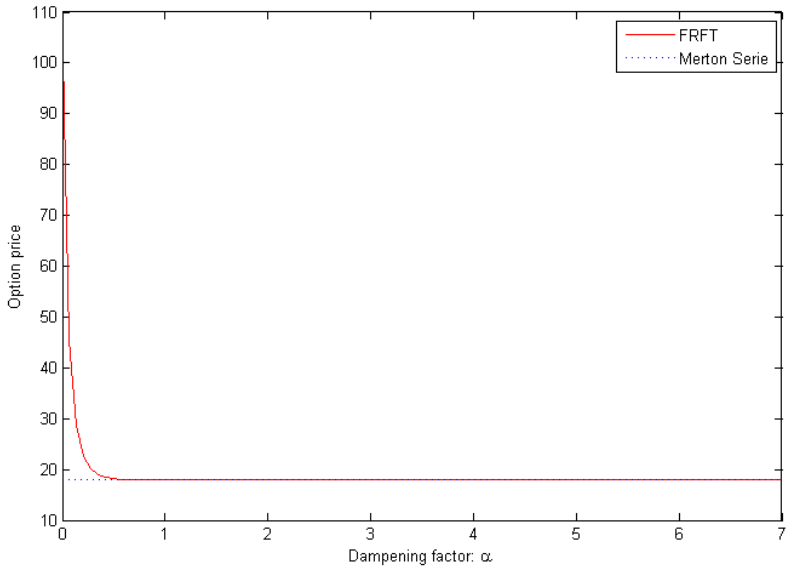



Figure 5.2: The value of the option price for different values of the dampening factor $\alpha$. The parameters are chosen to be S=110, K=100, r=0.05, T=1, sigma=0.2, lambda=0.1, jumpa=-0.1, jumpb=0.2

FRFT approximates satisfactorily the "true" option price (computed by the Merton series expansion). In general, we found that a value of $\alpha$ chosen in the interval $[2, 5]$ gives very good results.

### 5.4.2 The Fourier Approximation

Since we have a closed-form solution for the Merton Jump-Diffusion model (the Merton series expansion) we can compare the results obtained by using this formula with the results obtained by using FRFT. This comparison shows where the FRFT method does not work well.

Let's first describe the Merton series expansion. Assume we have a Merton Jump-Diffusion with parameters $\sigma$, $\lambda$, $\alpha$ and $\delta$ and $r$ is the risk-free interest rate, $\tau$ the time to maturity, $S$ the spot price and $K$ the strike price. If we set

$$\lambda' = \lambda e^{\alpha + \frac{1}{2}\delta^2} \tag{5.38}$$

and for $i = 0, \ldots, n$ with $n \to \infty$

$$\sigma_i' = \left(\sigma^2 + \frac{i}{\tau}\delta^2\right)^{1/2} \tag{5.39}$$

$$r_i' = r + \frac{i}{\tau}\left(\alpha + \frac{1}{2}\delta^2\right) - \lambda\left(e^{\alpha + \frac{1}{2}\delta^2} - 1\right) \tag{5.40}$$

Analogously

$$f_i(S, K, \tau) = BS(S, K, \tau, r_i, \sigma_i) \tag{5.41}$$

where $BS$ is the Black-Scholes price.

Then, the option price given by the Merton Jump-Diffusion is given by

$$C_{JD}(S, K, \tau, r, \sigma, \lambda, \alpha, \delta) = \sum_{i=0}^{n} \frac{e^{-\lambda'\tau}(\lambda'\tau)^i}{i!} f_i(S, K, \tau) \tag{5.42}$$

Ideally, $n$ has to go to infinity but we found that with a number for $n$ superior to 10 (say 30 in average) we obtain good results.

The code in MATLAB that performs the Merton Series Expansion can be found in appendix.

Now we have a closed-form result for the jump-diffusion model, we can confront it with the result given by the FRFT procedure and test the goodness of the method (above all because we need to use this procedure for models for which a closed-form doesn't exist). We use the *Average Relative Percentage Error* given by

$$ARPE = \frac{1}{N}\sum_{i=0}^{N}\frac{|P_{MSE} - P_{FRFT}|}{P_{MSE}} \tag{5.43}$$

where $N$ is the number of options prices we used, $P_{MSE}$ is the price given by the Merton Series Expansion and $P_{FRFT}$ is the price given by the FRFT algorithm.

From the results in the table, we can clearly conclude that the error produced by the Fourier approximation is very little. The only cases where Fourier method gives unsatisfactory results are when the maturity tends to zero (very short maturities) and when the options are deep-in-the-money or deep-out-of-the-money. We are mainly interested in options around the strike price ($\pm 20\%$ of the ATM strike price) and this problem is not so important. For options near to the maturity, the Fourier method cannot give us satisfactory prices, so we will take into account this fact for the calibration of the model and we use options with at least 1-month maturity.

```
ARPE (Average Relative Percentage Error) :
   Number of points : 100
   Parameters values :
S = 1.00; K = 1.00; r = 0.05; T = 1.00; sigma = 0.15;
lambda = 0.10; jumpa = -0.05; jumpb = 0.20

S_ATM    :   +3.799399e-009     T        :   +3.472825e-014
S_ITM    :   +2.130608e-010     T_near0  :   +1.484451e+001
S_DITM   :   +3.521305e-012     T_infty  :   +5.048629e-016
S_OTM    :   +2.231980e-008     r        :   +3.794329e-016
S_DOTM   :   +1.012594e+035     sigma    :   +4.480838e-016
K_ATM    :   +1.196277e-009     lambda   :   +4.518968e-016
K_ITM    :   +2.146435e-010     jumpa_n  :   +4.592392e-016
K_DITM   :   +5.480871e-004     jumpa_p  :   +5.299982e-016
K_OTM    :   +1.467181e-008     jumpb    :   +5.180194e-016
K_DOTM   :   +4.918132e-008
```

In figure 5.4.2 we plot the Average Relative Percentage Error for various parameters. On the X axis we plot a range of values of the parameter (for example, Spot ATM represents $\pm 25\%$ of the strike price).

### 5.4.3 The Implied Volatility Surface

Using a much shorter input vector, FRFT is faster than FFT and allows to calculate simultaneously option prices for various maturities. Then using an iterative search procedure like the Newton-Raphson method, we can easily find the implied volatilities for every option prices and plot an implied volatility surface.

In figure 5.4.3 we plot the volatility surfaces generated with four models: Merton Jump-Diffusion, Kou Double-Exponential Jump-Diffusion, Variance Gamma and Normal Inverse Gaussian.

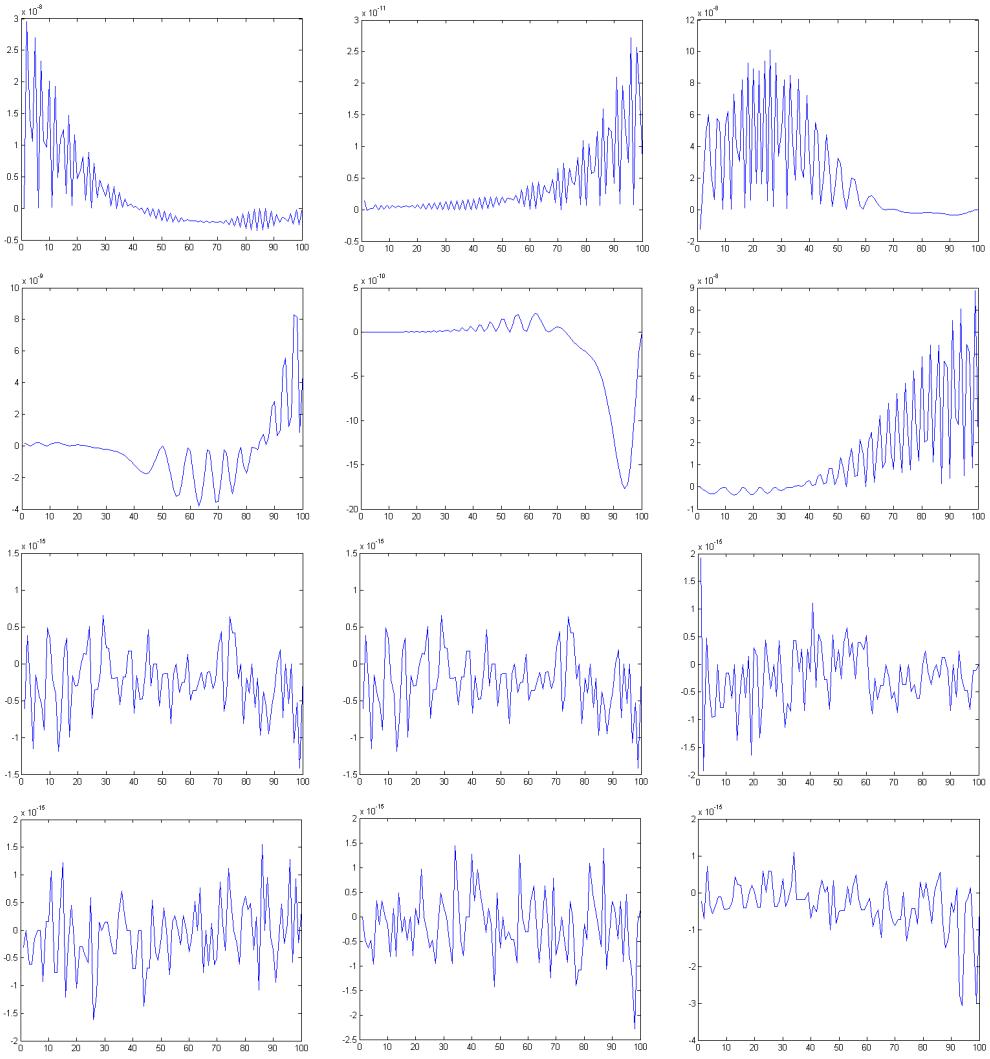


Figure 5.3: Relative errors versus parameters ranging around the following: $S = 1$, $K = 1$, $r = 0.05$, $T = 1$, $\sigma = 0.15$, $\lambda = 0.1$, $\alpha = -0.05$ and $\delta = 0.2$. From top-left to bottom-right: Spot price ATM ($[80\%, 120\%]$ of $S$), Spot price ITM ($[100\%, 140\%]$ of $S$), Spot price OTM ($[60\%, 100\%]$ of $S$), Strike price ATM ($[80\%, 120\%]$ of $K$), Strike price ITM ($[60\%, 100\%]$ of $K$), Strike price OTM ($[100\%, 140\%]$ of $K$), Interest rate ($[80\%, 120\%]$ of $r$), Volatility ($[80\%, 120\%]$ of $\sigma$), Jumps/year ($[80\%, 120\%]$ of $\lambda$), Mean jump ($[80\%, 120\%]$ of $\alpha$), Jump volatility ($[80\%, 120\%]$ of $\delta$)

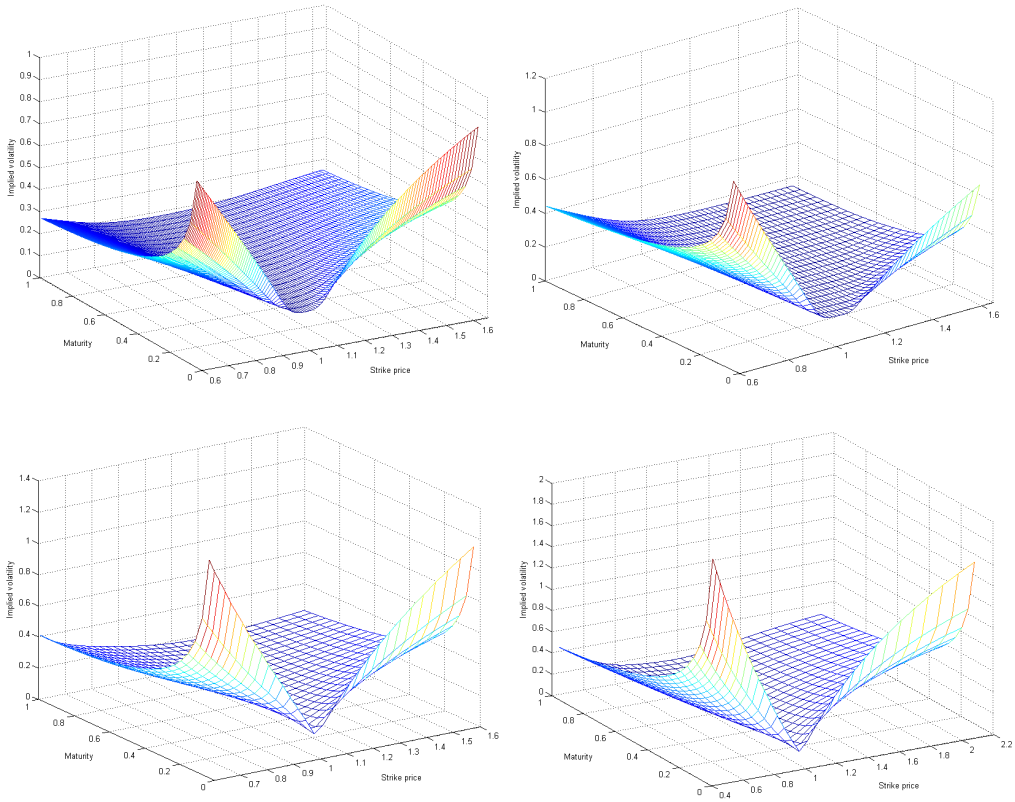



Figure 5.4: Smiles surfaces generated with the Jump-Diffusion model ($r = 0.05, \lambda = 0.1, \sigma = 0.15, a = -0.05, b = 0.4$), Double-Exponential model ($r = 0.05, \lambda = 0.05, p = 0.09, \eta_1 = 0.25, \eta_2 = 0.4$), NIG model ($r = 0.05, \delta = 0.1622, \alpha = 5.1882, \beta = -2.8941$) and VG model ($r = 0.05, \sigma = 0.1213, \theta = -0.1436, \nu = 0.1686$)

# Chapter 6

# Calibration

Now we are able to simulate the path of a Lévy process, to obtain option prices under it, the next step is to calibrate this model to market option prices. The calibration consists in estimating the (unknown) parameters of our model which reproduce (almost) perfectly the market option prices. This object is the analog, for exponential Lévy models, of implied volatility, used in the Black-Scholes framework. The main purpose of calibration is pricing OTC options, often exotic, which don't quote in any market and which prices are therefore unknown. For this purpose, to calibrate the model, we use vanilla option prices, which are quoted on some exchange.

We first explain the calibration procedure and in a second part we expose the results obtained.

All the data in this chapter have been entirely downloaded (freely) from the following websites:

Bloomberg (`http://www.bloomberg.com`)

MSN Money (`http://moneycentral.msn.com`)

Reuters (`http://www.reuters.com`)

The Wall Street Journal (http://www.wsj.com)
Yahoo! Finance (http://finance.yahoo.com)
The Options Industry Council (http://www.optionseducation.org).

## 6.1 The Calibration Inputs

To realize the model calibration, we need option market prices. Among the large choice of options available on the market, we select index options. The main reason is that index options, unlike stock options, are European-style. The pricing methods we develop in the former chapters work only with European-style options. We could extend these models to American-Style options, but this would be beyond the scope of this thesis. Recently, with the boom of ETFs (Exchange Trade Funds), classic index products become less liquid. As an example, nowadays, some traders assume to watch QQQQ quotes (an ETF tracking the Nasdaq-100 Index) as a benchmark for Nasdaq market. Analogously, options on ETFs are much more liquid than index options. However, an ETF share being sort of a stock, ETF options are American-style. As we need European-style options, we choose Index options. Among them, the far most liquid is SPX, an option on the Standard & Poors 500 Index. The second most liquid product is RUT, an option on the Russell-2000 Index. Another choices are DJX (on the Dow Jones Industrial Average Index), NDX (on the Nasdaq-100 Index) and XEO (the European-style alternative to OEX, famous option on the Standard & Poors 100 Index).
For the purpose of calibration, we decide to use options at various strikes and maturities on SPX, RUT, DJX and NDX.
We cannot consider option pricing without talking about interest rates and

| Bills | | Maturity Date | Price | Yield |
|---|---|---|---|---|
| 1 month | | Mar-27-2008 | 99.87 | 2.030 |
| 3 month | | May-29-2008 | 99.60 | 1.705 |
| 6 month | | Aug-28-2008 | 99.12 | 1.810 |
| Notes/Bonds | Coupon | Maturity Date | Price | Yield |
| 2 year | 2.000 | Feb-28-2010 | 100.76 | 1.612 |
| 5 year | 2.750 | Feb-28-2013 | 101.27 | 2.477 |
| 10 year | 3.500 | Feb-15-2018 | 99.72 | 3.534 |
| 30 year | 4.375 | Feb-15-2038 | 99.19 | 4.424 |

Table 6.1: U.S. Treasury Bonds Rates on 03-March-08

dividends. These two factors are input in our option pricing models and have to be accurately chosen. As we are dealing with various maturities, the former cannot be considered constant in our model and we have to construct the famous yield curve. The latter, that is to say the expected dividend, being an expected value is not directly available in the market and has to be retrieved from futures quotes.

We first talk about interest rates, then, about the option sets and finally about dividends.

### 6.1.1 The Risk-Free Interest Rate

One difficulty that often arises in practice is how to derive the proper interest rate to use as an input. Though a truly risk-free asset exists only in theory, in practice most professionals and academics use short-dated government bonds of the currency in question. For USD investments, usually U.S. Treasury bills, notes and bonds are used. These securities are considered to be risk-free because the likelihood of the government defaulting is extremely low. However, since we are dealing with options over various maturities, we can't use the same

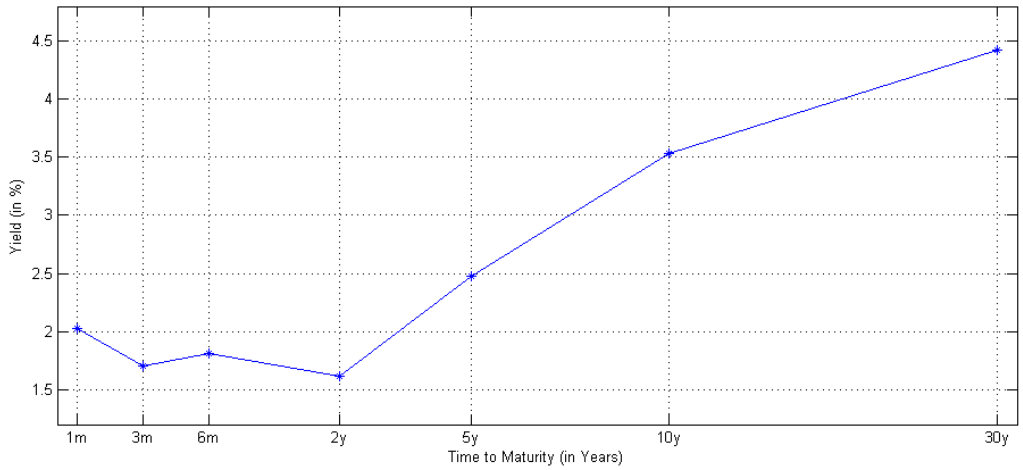



Figure 6.1: U.S. Treasury Yield Curve on 03-March-08

interest rate among all the maturities. The deposit rate for a risk-free bond maturing on the option's expiration date is, in general, not observable in the market. Instead, an interest rate curve is used. Therefore, we construct the U.S. Treasury yield curve interpolating (often done via bootstrapping) the yields all over the curve. The prices and corresponding yields of the bonds used to build the yield curve are in Table (6.1) and the yield curve is plotted on Figure (6.1). Then the curve provides an estimate of the risk-free rate of appropriate maturity for the option being priced. This technique allows to draw a yield curve with maturities until 30 years.

However, for option pricing use, we only need maturities until 3 years and above all with high precision. For this purpose we use Treasury-bill yield quotes and Short-Term OTC Treasury Notes. We wrote the yield quotes of the U.S. Treasury Bills and OTC Notes in Table (6.2) and we plotted the short-term high-frequency yield curve in Figure (6.2).

| Bills | | Notes | | Notes | |
|---|---|---|---|---|---|
| Maturity | Yield | Maturity | Yield | Maturity | Yield |
| 06-Mar-08 | 1.65 | 15-Sep-08 | 1.87 | 15-Oct-09 | 1.53 |
| 13-Mar-08 | 2.11 | 30-Sep-08 | 1.72 | 31-Oct-09 | 1.57 |
| 20-Mar-08 | 2.17 | 15-Oct-08 | 1.73 | 15-Nov-09 | 1.54 |
| 27-Mar-08 | 2.08 | 31-Oct-08 | 1.69 | 30-Nov-09 | 1.58 |
| 03-Apr-08 | 2.06 | 15-Nov-08 | 1.72 | 15-Dec-09 | 1.54 |
| 10-Apr-08 | 2.03 | 30-Nov-08 | 1.59 | 31-Dec-09 | 1.60 |
| 17-Apr-08 | 2.02 | 15-Dec-08 | 1.60 | 15-Jan-10 | 1.57 |
| 24-Apr-08 | 1.78 | 31-Dec-08 | 1.66 | 31-Jan-10 | 1.60 |
| 01-May-08 | 1.74 | 15-Jan-09 | 1.68 | 15-Feb-10 | 1.59 |
| 08-May-08 | 1.73 | 31-Jan-09 | 1.65 | 28-Feb-10 | 1.61 |
| 15-May-08 | 1.72 | 15-Feb-09 | 1.63 | 15-Mar-10 | 1.59 |
| 22-May-08 | 1.72 | 28-Feb-09 | 1.61 | 15-Apr-10 | 1.62 |
| 29-May-08 | 1.72 | 15-Mar-09 | 1.60 | 15-May-10 | 1.61 |
| 05-Jun-08 | 1.71 | 31-Mar-09 | 1.56 | 15-Jun-10 | 1.62 |
| 12-Jun-08 | 1.72 | 15-Apr-09 | 1.55 | 15-Jul-10 | 1.64 |
| 19-Jun-08 | 1.78 | 30-Apr-09 | 1.52 | 15-Aug-10 | 1.68 |
| 26-Jun-08 | 1.75 | 15-May-09 | 1.51 | 15-Sep-10 | 1.69 |
| 03-Jul-08 | 1.78 | 31-May-09 | 1.52 | 15-Oct-10 | 1.72 |
| 10-Jul-08 | 1.80 | 15-Jun-09 | 1.49 | 15-Nov-10 | 1.73 |
| 17-Jul-08 | 1.80 | 30-Jun-09 | 1.51 | 15-Dec-10 | 1.76 |
| 24-Jul-08 | 1.80 | 15-Jul-09 | 1.49 | 15-Jan-11 | 1.76 |
| 31-Jul-08 | 1.81 | 31-Jul-09 | 1.51 | 15-Feb-11 | 1.79 |
| 07-Aug-08 | 1.80 | 15-Aug-09 | 1.51 | 28-Feb-11 | 1.82 |
| 14-Aug-08 | 1.81 | 31-Aug-09 | 1.52 | 31-Mar-11 | 1.85 |
| 21-Aug-08 | 1.81 | 15-Sep-09 | 1.53 | 30-Apr-11 | 1.90 |
| 28-Aug-08 | 1.81 | 30-Sep-09 | 1.55 | 31-May-11 | 1.93 |

Table 6.2: U.S. Treasury Bill and OTC Note Rates on 13-Sep-07

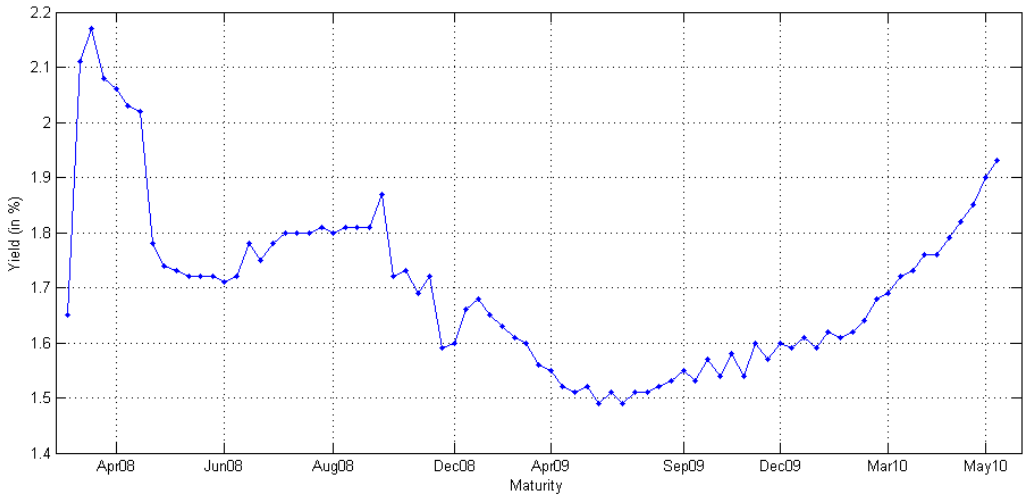



Figure 6.2: U.S. Treasury Yield Curve on 03-Mar-08 based upon Bills and Short-Term OTC Notes

### 6.1.2 The Options Sets

As said before, we chose four index option sets, on the Standard & Poors 500 Index (SPX), on the Russell-2000 Index (RUT), on the Dow Jones Industrial Average Index (DJX) and on the Nasdaq-100 Index (NDX). Among all the options available for each index, for liquidity reason, just a few are well priced, that is to say whose price reflects the real supply and demand in the market. Indeed, Deep-in-the-Money or Deep-Out-of-the-money options are often illiquid or with a price next to zero. For this reason, we have to make a selection among all the option set. We base our choices upon criterion we expose in following:

1. Options picked have at least one-month maturity (price of options near to the maturity is very close to the intrinsic value)

2. We pick only options close to the At-the-Money strike price, that is to say on the interval $[80\%, 120\%]$ of the ATM strike price

3. Options for which the expression $\frac{Ask\ price\ -\ Bid\ price}{(Bid\ price\ +\ Ask\ price)/2}$ is greater than 10% are eliminated (a too large Bid-Ask spread often means a wrongly priced option due to little volume)

4. Options are chosen only if the implied volatility (IV) can be calculated (Deep-ITM options (close to intrinsic value) and Deep-OTM options (close to zero) often have unrealistic implied volatilies which cannot even be calculated). As we are interested in calibrating the implied volatility surface, we have to get IV for all option prices.

Now we know which type of options we are looking, let's take a look more precisely at the option sets. To select the option set, we just took a picture of the market during mid-afternoon of March 03, 2008. On this date, we picked up simultaneously (exactly at the same moment) SPX, RUT, DJX and DJX quotes, all the options prices available on this day for these indexes, U.S. Treasury bill and bond prices (to calculate the bond yield curve for interest risk-free rate) and index futures quotes for calculating expected dividends. Out of the more than 600 SPX Call options available on the market on March 03, 2008, only 80 are conform with our requisites; 96 on RUT, 154 on DJX and 84 on NDX.

### Standard & Poors 500 Index: SPX

The first option set is taken on the Standard & Poors 500 Index. Widely regarded as the best single gauge of the U.S. equities market, this world-renowned index includes 500 leading companies in leading industries of the U.S. economy. Standard & Poors 500 Index is a very liquid index, often highly volatile. To confirm this feature, we insert a chart, see Figure (6.3), plotting the last ten trading day quotes for the Standard & Poors 500 Index. Indeed, we can clearly denote a high volatility (with a few price gaps at beginning of trading days). In Table (6.3) we represent the option prices for various maturities and strike prices available for SPX on March 03, 2008. As said before, a significant number of options have been eliminated and we keep the ones which correspond to our criterion.

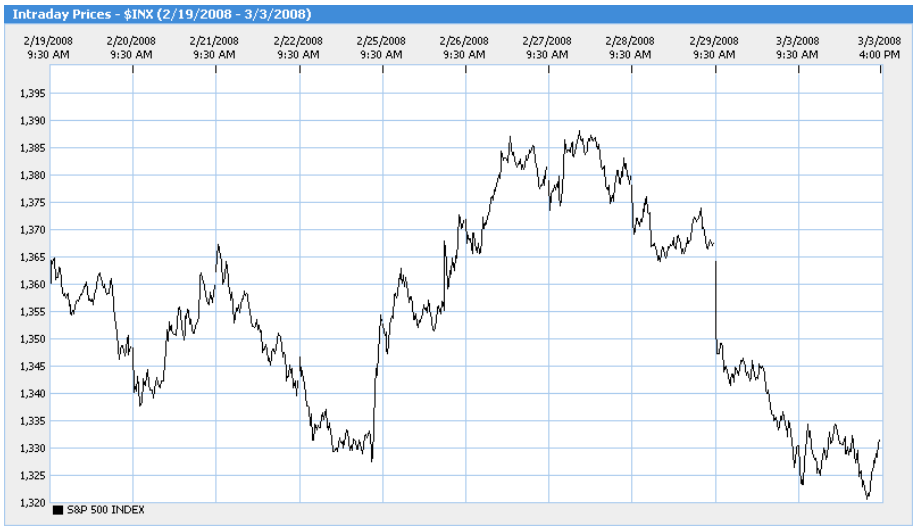



Figure 6.3: Standard & Poors 500 Index (SPX) prices on the last 10 trading days.

| Strike | S/K | Time to maturity (in days) | | | | |
|---|---|---|---|---|---|---|
| | | 47 | 75 | 110 | 201 | 292 |
| 1100 | 1.21 | 231.60 | 235.70 | 242.80 | 256.40 | 267.60 |
| 1150 | 1.15 | 185.20 | 191.30 | 200.10 | 216.50 | 229.50 |
| 1175 | 1.13 | 162.80 | 170.00 | 179.60 | 197.40 | 211.30 |
| 1200 | 1.11 | 141.10 | 149.40 | 159.80 | 178.90 | 193.60 |
| 1225 | 1.08 | 120.30 | 129.70 | 140.70 | 161.10 | 176.40 |
| 1250 | 1.06 | 100.60 | 110.90 | 122.60 | 144.00 | 160.00 |
| 1275 | 1.04 | 82.10 | 93.30 | 105.30 | 127.60 | 144.10 |
| 1300 | 1.02 | 65.10 | 76.90 | 89.20 | 112.00 | 129.00 |
| 1325 | 1.00 | 49.80 | 61.90 | 74.20 | 97.40 | 114.60 |
| 1350 | 0.98 | 36.35 | 48.10 | 60.30 | 83.60 | 100.90 |
| 1375 | 0.97 | 24.90 | 36.20 | 47.90 | 70.80 | 88.20 |
| 1400 | 0.95 | 15.90 | 26.10 | 37.00 | 59.20 | 76.40 |
| 1425 | 0.93 | 9.45 | 18.00 | 27.80 | 48.80 | 65.50 |
| 1450 | 0.92 | 4.90 | 11.80 | 20.20 | 39.60 | 55.70 |
| 1475 | 0.90 | 2.40 | 7.10 | 14.10 | 31.60 | 46.80 |
| 1500 | 0.88 | 1.08 | 4.20 | 9.50 | 24.70 | 38.90 |

Table 6.3: SPX option prices


Figure 6.4: Russell-2000 Index (RUT) prices on the last 10 trading days.

**Russell-2000 Index: RUT**

The Russell-2000 Index measures the performance of the small-cap segment of the U.S. equity universe. It includes approximately 2000 of the smallest securities based on a combination of their market cap and current index membership. The Russell 2000 is constructed to provide a comprehensive and unbiased small-cap barometer and is completely reconstituted annually to ensure larger stocks do not distort the performance and characteristics of the true small-cap opportunity set. Due to these features, this index used to be very volatile.

We include a chart, see Figure (6.4), plotting the last ten trading day quotes of the Russell-2000 Index. In Table (6.4) we represent the option prices for various maturities and strike prices available for RUT on March 03, 2008.

| Strike | S/K | Time to maturity (in days) | | | |
|---|---|---|---|---|---|
| | | 47 | 75 | 110 | 201 |
| 550 | 1.24 | 132.05 | 135.25 | 139.05 | 147.60 |
| 560 | 1.22 | 122.75 | 126.45 | 130.65 | 139.60 |
| 570 | 1.19 | 113.65 | 117.80 | 122.45 | 131.95 |
| 580 | 1.17 | 104.60 | 109.40 | 114.40 | 124.50 |
| 590 | 1.15 | 95.90 | 101.10 | 106.50 | 117.35 |
| 600 | 1.13 | 87.40 | 93.15 | 98.90 | 110.25 |
| 610 | 1.12 | 79.10 | 85.30 | 91.45 | 103.45 |
| 620 | 1.10 | 71.00 | 77.75 | 84.15 | 96.75 |
| 630 | 1.08 | 63.30 | 70.40 | 77.25 | 89.95 |
| 640 | 1.06 | 55.95 | 63.40 | 70.45 | 83.50 |
| 650 | 1.05 | 48.95 | 56.70 | 63.95 | 77.25 |
| 660 | 1.03 | 42.35 | 50.25 | 57.55 | 71.65 |
| 670 | 1.02 | 36.15 | 44.25 | 51.65 | 65.55 |
| 680 | 1.00 | 30.45 | 38.50 | 46.05 | 60.25 |
| 690 | 0.99 | 25.20 | 33.25 | 40.90 | 54.95 |
| 700 | 0.97 | 20.55 | 28.30 | 35.80 | 49.80 |
| 710 | 0.96 | 16.40 | 23.90 | 31.25 | 45.25 |
| 720 | 0.95 | 12.85 | 20.05 | 27.10 | 40.65 |
| 730 | 0.93 | 9.85 | 16.45 | 23.25 | 36.55 |
| 740 | 0.92 | 7.35 | 13.40 | 19.75 | 32.65 |
| 750 | 0.91 | 5.35 | 10.70 | 16.75 | 29.05 |
| 760 | 0.90 | 3.80 | 8.50 | 14.05 | 25.70 |
| 770 | 0.88 | 2.60 | 6.60 | 11.55 | 22.55 |
| 780 | 0.87 | 1.75 | 5.10 | 9.55 | 19.75 |

Table 6.4: RUT option prices

### Nasdaq-100 Index: NDX

The Nasdaq-100 Index, widely perceived as a technology benchmark, includes 100 of the largest domestic and international non-financial securities listed on the Nasdaq Stock Market based on market capitalization. The Index reflects companies across major industry groups including computer hardware and software, telecommunications, retail/wholesale trade and biotechnology. It does not contain securities of financial companies including investment companies.

In Figure (6.5) we plot the last 10 trading day quotes and volumes of the Nasdaq-100 Index. As it can be seen, the Nasdaq-100 Index is highly volatile. In Table (6.5) we listed the option quotes for various strike prices and maturities.

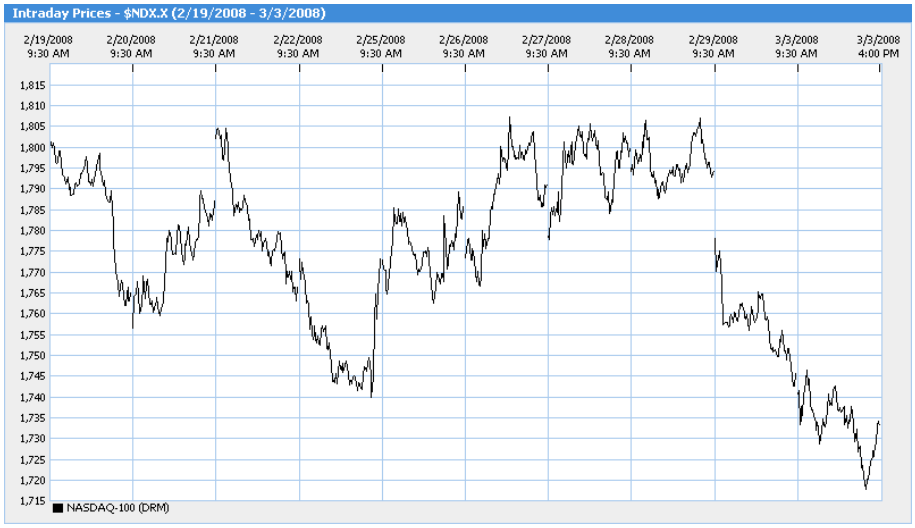



Figure 6.5: Nasdaq-100 Index (NDX) prices on the last 10 trading days.

| Strike | S/K | Time to maturity (in days) | | | |
|---|---|---|---|---|---|
| | | 47 | 75 | 110 | 201 |
| 1400 | 1.23 | 334.95 | 343.40 | 354.90 | 378.80 |
| 1425 | 1.21 | 311.55 | 321.10 | 333.15 | 359.95 |
| 1450 | 1.19 | 288.35 | 298.90 | 311.55 | 340.15 |
| 1475 | 1.17 | 265.50 | 277.20 | 290.85 | 320.90 |
| 1500 | 1.15 | 242.60 | 255.90 | 270.25 | 302.05 |
| 1525 | 1.13 | 220.50 | 235.15 | 250.45 | 283.70 |
| 1550 | 1.11 | 198.95 | 215.05 | 231.50 | 265.65 |
| 1575 | 1.10 | 178.45 | 195.10 | 212.70 | 248.10 |
| 1600 | 1.08 | 158.55 | 176.55 | 194.60 | 231.15 |
| 1625 | 1.06 | 139.60 | 158.35 | 177.00 | 214.60 |
| 1650 | 1.05 | 121.50 | 140.95 | 160.35 | 198.65 |
| 1675 | 1.03 | 104.45 | 124.45 | 144.30 | 183.35 |
| 1700 | 1.02 | 88.45 | 108.95 | 128.95 | 168.45 |
| 1725 | 1.00 | 73.80 | 94.35 | 114.40 | 153.85 |
| 1750 | 0.99 | 60.40 | 80.80 | 100.75 | 140.30 |
| 1775 | 0.97 | 48.45 | 68.20 | 87.90 | 127.40 |
| 1800 | 0.96 | 38.05 | 57.20 | 76.10 | 115.45 |
| 1825 | 0.95 | 29.20 | 47.00 | 65.60 | 103.80 |
| 1850 | 0.93 | 21.65 | 38.35 | 55.55 | 93.10 |
| 1875 | 0.92 | 15.65 | 30.45 | 46.80 | 83.15 |
| 1900 | 0.91 | 10.95 | 24.00 | 38.95 | 73.75 |
| 1925 | 0.90 | 7.45 | 18.50 | 32.00 | 65.20 |

Table 6.5: NDX option prices

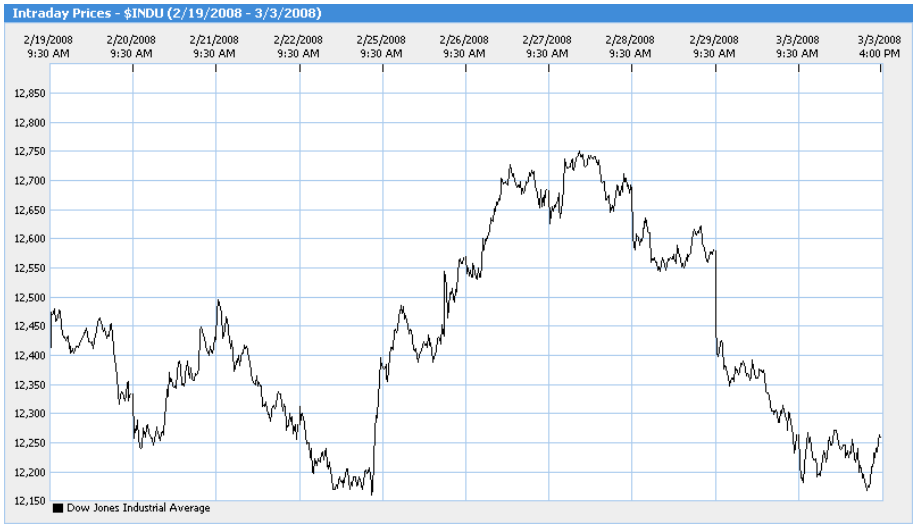



Figure 6.6: Dow Jones Industrial Average Index (DJX) prices and volume on the last 10 trading days.

### Dow Jones Industrial Average Index: DJX

The 30 stocks in the Dow Jones Industrial Average Index are all major factors in their industries, and their stocks are widely held by individuals and institutional investors. Founded in 1896, the DJIA index accounts for approximately 23.8% of the total U.S. market. The Dow Jones Industrial Average Index is the most-quoted market indicator in newspapers, on TV and on the Internet. Because of its longevity, it became the first to be quoted by other publications.

We include a chart (see Figure (6.6)) plotting the last ten trading day quotes of the Dow Jones Industrial Average Index. On Table (6.6) we listed option quotes on DJX security.

| Strike | S/K | Time to maturity (in days) | | | | |
|---|---|---|---|---|---|---|
| | | 47 | 75 | 110 | 201 | 292 |
| 98 | 1.25 | 24.43 | 24.60 | 24.95 | 25.88 | 26.63 |
| 99 | 1.23 | 23.40 | 23.63 | 24.10 | 25.03 | 25.83 |
| 100 | 1.22 | 22.50 | 22.73 | 23.20 | 24.18 | 25.03 |
| 101 | 1.21 | 21.55 | 21.80 | 22.30 | 23.35 | 24.20 |
| 102 | 1.20 | 20.63 | 20.88 | 21.40 | 22.53 | 23.40 |
| 103 | 1.19 | 19.68 | 20.08 | 20.55 | 21.70 | 22.68 |
| 104 | 1.18 | 18.75 | 19.15 | 19.70 | 20.90 | 21.88 |
| 105 | 1.16 | 17.83 | 18.25 | 18.85 | 20.10 | 21.13 |
| 106 | 1.15 | 16.90 | 17.30 | 18.00 | 19.30 | 20.38 |
| 107 | 1.14 | 15.98 | 16.53 | 17.15 | 18.53 | 19.63 |
| 108 | 1.13 | 15.10 | 15.60 | 16.33 | 17.75 | 18.90 |
| 109 | 1.12 | 14.23 | 14.75 | 15.50 | 17.03 | 18.18 |
| 110 | 1.11 | 13.33 | 13.98 | 14.70 | 16.28 | 17.35 |
| 111 | 1.10 | 12.45 | 13.10 | 13.93 | 15.53 | 16.65 |
| 112 | 1.09 | 11.63 | 12.30 | 13.18 | 14.80 | 16.00 |
| 113 | 1.08 | 10.78 | 11.50 | 12.40 | 14.10 | 15.35 |
| 114 | 1.07 | 9.95 | 10.80 | 11.65 | 13.40 | 14.63 |
| 115 | 1.06 | 9.18 | 9.98 | 10.93 | 12.70 | 13.98 |
| 116 | 1.05 | 8.40 | 9.30 | 10.23 | 12.03 | 13.33 |
| 117 | 1.04 | 7.68 | 8.58 | 9.53 | 11.35 | 12.73 |
| 118 | 1.04 | 6.93 | 7.85 | 8.83 | 10.73 | 12.05 |
| 119 | 1.03 | 6.23 | 7.20 | 8.18 | 10.10 | 11.45 |
| 120 | 1.02 | 5.58 | 6.58 | 7.55 | 9.50 | 10.85 |
| 121 | 1.01 | 4.95 | 5.93 | 6.93 | 8.90 | 10.33 |
| 122 | 1.00 | 4.35 | 5.35 | 6.35 | 8.33 | 9.75 |
| 123 | 0.99 | 3.80 | 4.80 | 5.78 | 7.78 | 9.23 |
| 124 | 0.99 | 3.25 | 4.23 | 5.23 | 7.23 | 8.63 |
| 125 | 0.98 | 2.74 | 3.75 | 4.73 | 6.73 | 8.15 |
| 126 | 0.97 | 2.28 | 3.25 | 4.23 | 6.23 | 7.63 |
| 127 | 0.96 | 1.90 | 2.79 | 3.78 | 5.73 | 7.15 |
| 128 | 0.95 | 1.52 | 2.37 | 3.35 | 5.28 | 6.70 |

Table 6.6: DJX option prices

### 6.1.3 Dividends

We cannot talk about option pricing without talking about dividend. Stocks used to pay dividends. A dividend is a cash payment made to the owner of a stock. In this paragraph, we assume that the stock pays a continuous compound dividend yield at a rate $q$ per annum. Clearly, if our asset is an index, the dividend yield is the average of the dividend yields on the stocks comprising the index. A dividend can be paid once a year or more frequently. When a dividend is paid, the stock price goes down by the amount of the dividend. Quotes of already paid dividends are available on every financial website or newspaper. However, for option pricing, we need to quantify the amount of cash (dividend) that an owner of a stock can expect to receive until maturity. Clearly, being an expected value, it is not directly available on the

market. Some websites are expert in expected dividend calculation but we prefer not to use those, often too obscure, methods.
Rather, as futures quotes for various maturities are available on the market, we determine dividend yields from futures prices. For an asset with price $S_0$ paying a continuous (dividend) yield $q$ per annum, $r$ being the risk-free rate, the forward (future) price of that asset is then given by

$$F_0 = S_0 e^{(r-q)T} \tag{6.1}$$

where $T$ is the maturity of the forward (future) contract.
As $S_0$, $F_0$, $r$ and $T$ are directly available on the market, we can easily retrieve the expected dividend yields $q$ for various maturities. If option maturity falls between two future maturities, we simply calculate $q$ by interpolation.
In the following table (see Table (6.7)), we enter the future prices of the four indices for various maturities. In a second column, we give the respective free-risk interest rates for each maturity. Finally, in a third column, we can write the total discount factor to use for option pricing as a difference between the risk-free interest rate and the dividend rate.

## 6.2 The Black-Scholes Results

The main purpose of this thesis is to demonstrate that the classic Black-Scholes model fails in modeling option prices and that some Lévy processes, after accurate calibration, can almost perfectly reproduce the option prices and the volatility surface. The first task before calibrating the Lévy models is to price options for various strikes and maturities with the Black-Scholes model (using the historical volatility as an input) and compare them with the real prices

| | Date | Price | Dis. Fac. $(r-q)$ |
|---|---|---|---|
| SPX | Spot | 1327.16 | |
| | Apr08 | 1328.15 | 0.58% |
| | May08 | 1328.61 | 0.53% |
| | Jun08 | 1328.96 | 0.45% |
| | Sep08 | 1329.06 | 0.26% |
| | Dec08 | 1327.58 | 0.04% |
| RUT | Spot | 680.73 | |
| | Apr08 | 681.60 | 0.98% |
| | May08 | 682.02 | 0.91% |
| | Jun08 | 682.44 | 0.82% |
| | Sep08 | 683.09 | 0.62% |
| DJX | Spot | 122.22 | |
| | Apr08 | 122.34 | 0.74% |
| | May08 | 122.39 | 0.67% |
| | Jun08 | 122.45 | 0.61% |
| | Sep08 | 122.49 | 0.39% |
| | Dec08 | 122.40 | 0.18% |
| NDX | Spot | 1725.52 | |
| | Apr08 | 1731.14 | 2.49% |
| | May08 | 1734.31 | 2.44% |
| | Jun08 | 1738.17 | 2.39% |
| | Sep08 | 1747.14 | 2.23% |

Table 6.7: Discount factor (r-q) to use as an input for option pricing calculated from future prices.

| | 10 days | 20 days | 30 days |
|---|---|---|---|
| DJX | 18.99% | 19.49% | 21.03% |
| NDX | 18.47% | 22.95% | 23.22% |
| RUT | 23.80% | 26.24% | 27.12% |
| SPX | 19.46% | 20.65% | 22.30% |

Table 6.8: Historical volatilities used as an input in the Black-Scholes model.

observed in the market.

From the graphics, we can deduce that the Black-Scholes model clearly doesn't fit option market prices. The volatility used as an input is the historical volatility of the indexes during the last 10 days. We also used the historical volatility at 20 and 30 days (see table (6.8)) but the results are similar. We could calibrate the $\sigma$ parameter with option market prices but we still get a really bad fitting.

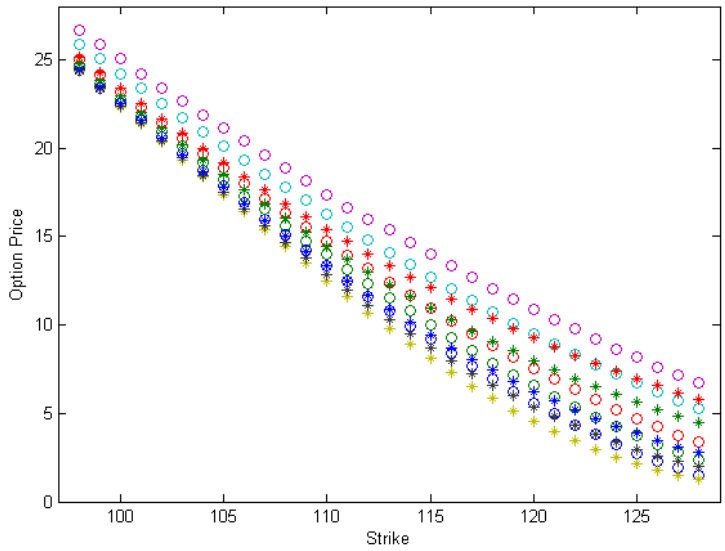



Figure 6.7: Option market prices on 03/03/2008 for DJX (rings) and prices given by the Black-Scholes model (stars).

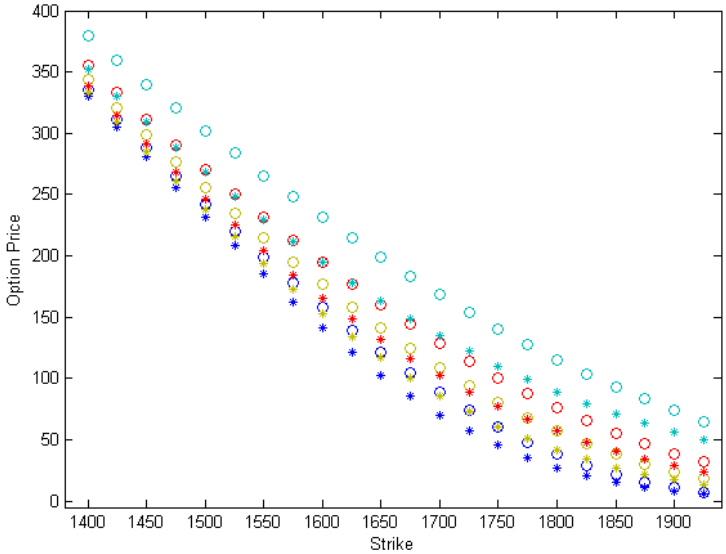



Figure 6.8: Option market prices on 03/03/2008 for NDX (rings) and prices given by the Black-Scholes model (stars).

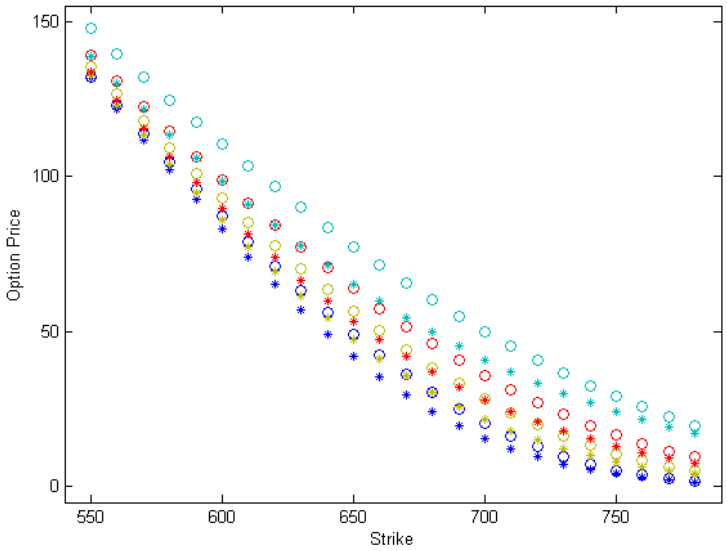



Figure 6.9: Option market prices on 03/03/2008 for RUT (rings) and prices given by the Black-Scholes model (stars).

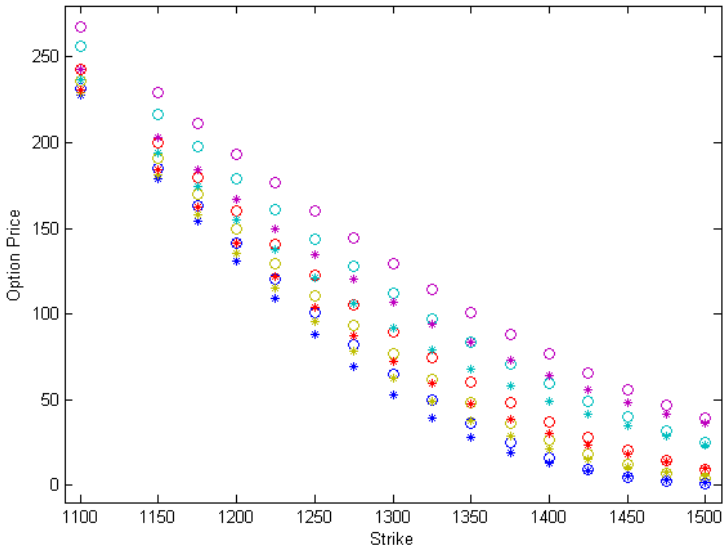



Figure 6.10: Option market prices on 03/03/2008 for SPX (rings) and prices given by the Black-Scholes model (stars).

## 6.3 Estimation Methods: Non-Linear Least-Squares

In the models we develop earlier, parameters estimation is not an easy task. The likelihood functions are not known in closed form for continuous-time Lévy models, since the observations are discrete. So the maximum likelihood method is very difficult to implement. One can use the underlying stock prices only to estimate the structural parameters. Indirect inference method is first proposed by Gourieroux, Monfort, and Renault (1993). This method is a simulation based moment matching method. Frequently used methods for Lévy models include the generalized method-of-moments (GMM) developed by Hansen and Scheinkman (1995), and the efficient method-of-moments (EMM) proposed by Gallant and Tauchen (1996). But in practice, it is not convenient to employ these econometric tools. An alternative and popular method is to use option prices directly. The procedure is simple. First, we collect $N$ options on the same stock in the same day. These options have different time to maturities and strike prices. Let $c_i^{Market}$ be the price of the $i$-th option, and $c_i^{Model}$ be its price determined by the model. The parameter set $\theta$ is then determined minimizing the following expression

$$\sum_{i=1}^{N} \left(c_i^{Market} - c_{i,\theta}^{Model}\right)^2 \tag{6.2}$$

that is to say, the objective consists of finding the minimum value of a sum of $N$ squared residuals (difference between market prices and prices obtained with the model) with respect to a set of n parameters of a model.

$$\widehat{\theta} = \arg\min_{\theta} \sum_{i=1}^{N} \left(c_i^{Market} - c_{i,\theta}^{Model}\right)^2 \tag{6.3}$$

### 6.3.1 Gradient Descent

Gradient descent is an optimization algorithm. To find a local minimum of a function using gradient descent, one takes steps proportional to the minus gradient (or the approximate gradient) of the function at the current point. The gradient descent method is based on the observation that if the real-valued function $f(x)$ is defined and differentiable in a neighborhood of a point $x_0$ then $f(x)$ decreases fastest if one goes from $x_0$ in the direction of the minus gradient of $f$ at $x_0$, that is:

$$x_1 = x_0 - \epsilon \nabla f(x_0). \tag{6.4}$$

Choosing $\epsilon > 0$ small enough whe have $f(x_0) \geq f(x_1)$. With this observation in mind, one starts with an initial guess $x_0$ for a local minimum of $f$, and generates the sequence $x_0, x_1, x_2, \ldots$ using the following equation:

$$x_{n+1} = x_n - \epsilon \nabla f(x_n), \qquad n = 0, 1, 2, \ldots. \tag{6.5}$$

When the gradient tends to zero, that means that the slope goes to zero and we get an approximation of a local minimum. The value of the step size $\epsilon$ is fixed previously but it is also allowed to change at every iteration deterministically or stochastically.

Obviously, in our case, the function $f$ to minimize is the difference between the market and model prices, and the point $x_0$ starting with is a vector containing a first guess of the model parameters. As we don't know the analytic form of the function $f$, we have to approximate its gradient with the finite-difference gradient. Thus, each component of the gradient can be approximate by the following expression

$$\frac{\partial f(x)}{\partial x} = \frac{f(x+h) - f(x)}{h} \tag{6.6}$$

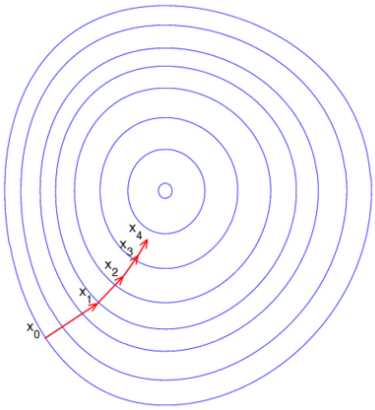



Figure 6.11: An example of the Gradient Descent procedure.

where $h > 0$ determines the step of the finite-difference gradient procedure. In the gradient descent algorithm, we thus have to determine two step sizes, $h$ which is, as explained before, the finite-difference procedure step, and $\epsilon$, which should be chosen larger than $h$, and represent the step size of the gradient descent procedure.

## 6.4 Results

For all the models we talked about in chapter 4, we tried to calibrate the parameters to the four index option sets (DJX, NDX, RUT and SPX). We fix the $h$ parameter low to get a good finite-differences approximation. The major problem is to fix the $\epsilon$ parameter and the tolerance, that is to say, the value we consider small enough to represent a minimum.

We look for initial parameters by plotting both market and model prices and trying manually to get a good fitting by varying the parameters. We use those initial parameters as a first guess. The best way to begin is to fix a very small

value for $\epsilon$ (around $1^{-7}$) and a quite big tolerance for the global square error. When the tolerance is reached, we update the initial parameters in the program with the new parameters, we decrease the tolerance and, step by step, we can also increase the value of $\epsilon$ .

With this procedure, we can get good fitting using all the models. However, since the gradient descent algorithm can easily find a local minimum, we cannot know if this minimum is also the global minimum. In that sense, the parameters we find with the algorithm are highly linked with the initial parameters we "guessed".

In Figures (6.14 - 6.19) we plotted the market and model prices (calculated with the parameters found after calibration) for all the strikes and maturities. It is difficult to say if a model works better than other. A model can work better on a dataset and worse on another. However, we found that NIG and VG models are easier to calibrate than JD and DE models since the global error decline rapidly (to reach $1e^{-4}$ for all the dataset) and all the parameters vary simultaneously from the initial ones during the calibration. In JD and DE cases, even when we start from initial parameters that seem adequate (from a graphical point of view), the global error often stops improving (never below $1e^{-3}$ for all the dataset) and one parameter moves significantly while the other ones keep their initial value or move slightly around. Probably this fact is imputable to the flatness of the objective function in that point.

The problems we encounter when calibrating the models are also due to the fact that we have a significant number of options for various strikes and maturities and the function to calibrate is thus quite tricky. When trying to

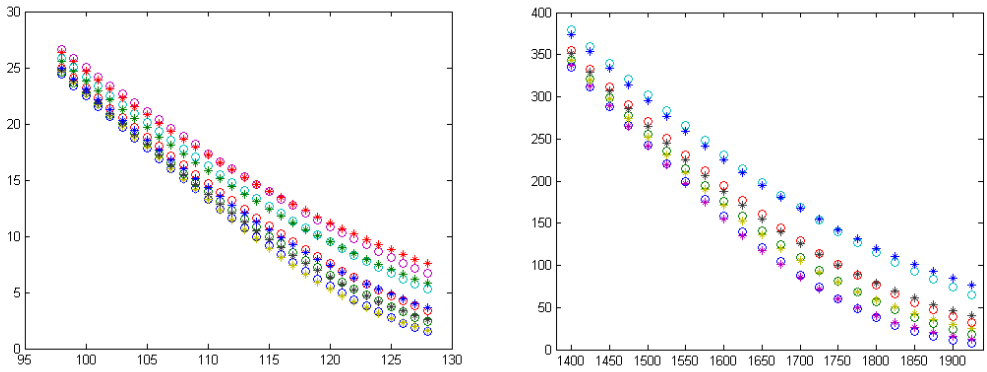



Figure 6.12: Option market prices on 03/03/2008 for DJX (left) and NDX (right) (rings) and prices given by the Jump-Diffusion model (stars).

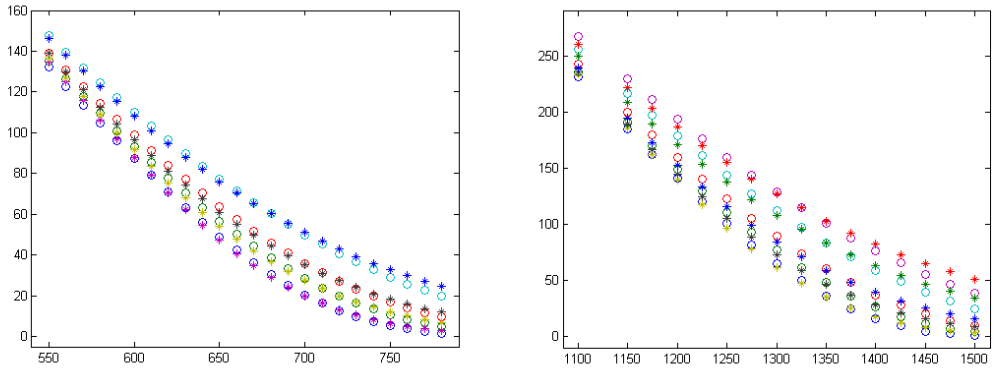



Figure 6.13: Option market prices on 03/03/2008 for RUT (left) and SPX (rings) and prices given by the Jump-Diffusion model (stars).

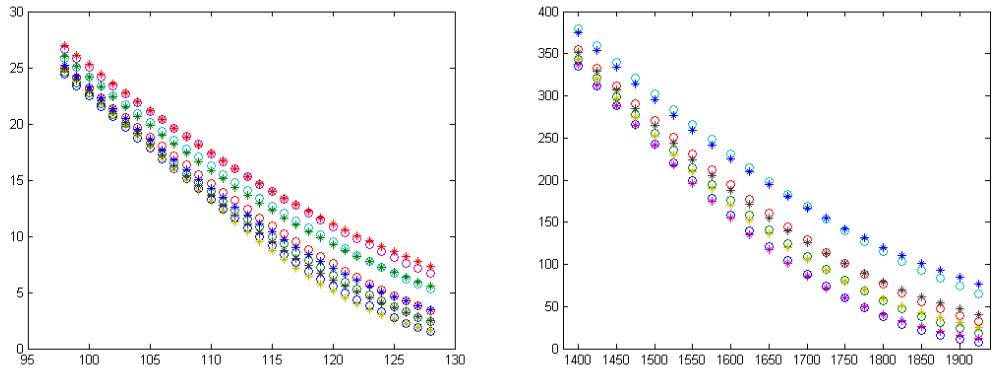



Figure 6.14: Option market prices on 03/03/2008 for DJX (left) and NDX (right) (rings) and prices given by the Double-Exponential model (stars).

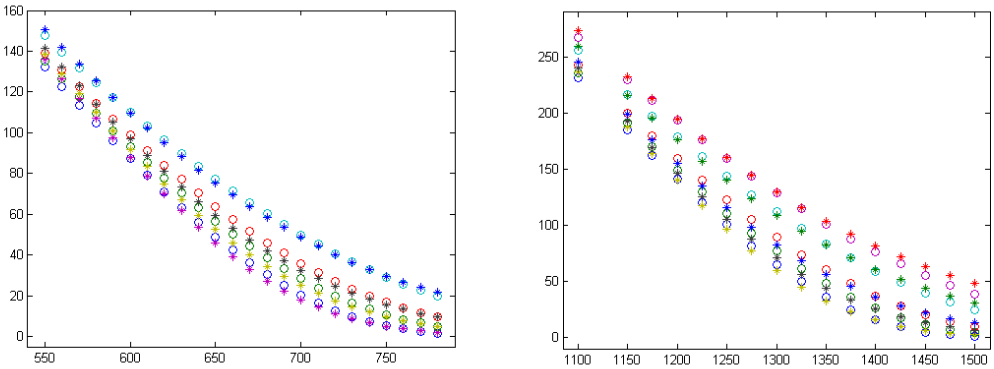



Figure 6.15: Option market prices on 03/03/2008 for RUT (left) and SPX (right) (rings) and prices given by the Double-Exponential model (stars).

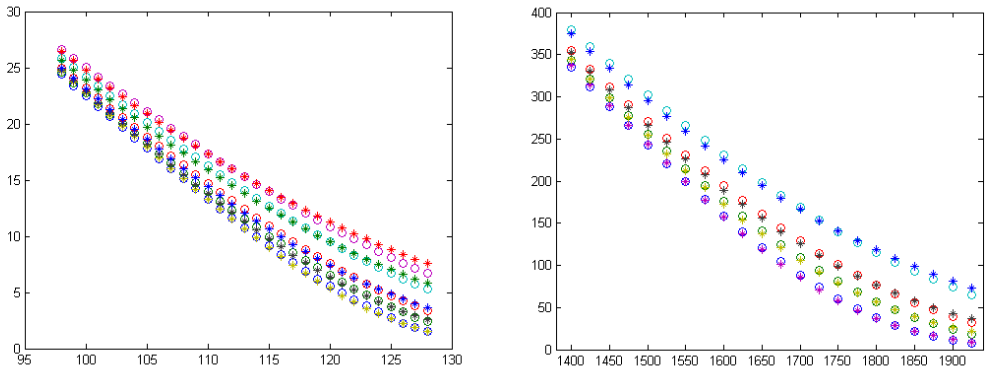



Figure 6.16: Option market prices on 03/03/2008 for DJX (left) and NDX (right) (rings) and prices given by the NIG model (stars).

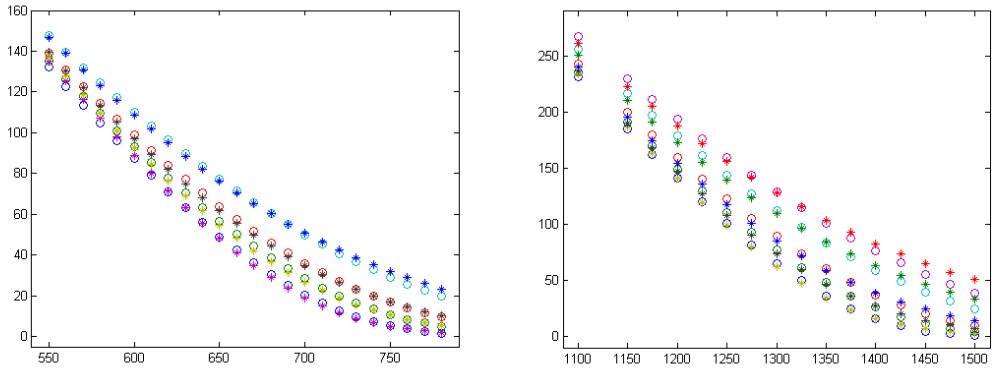



Figure 6.17: Option market prices on 03/03/2008 for RUT (left) and SPX (rings) and prices given by the NIG model (stars).

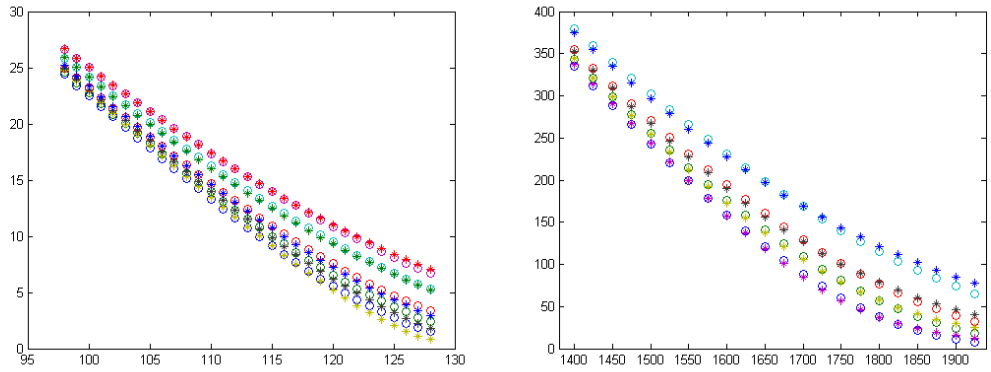



Figure 6.18: Option market prices on 03/03/2008 for DJX (left) and NDX (right) (rings) and prices given by the Variance-Gamma model (stars).

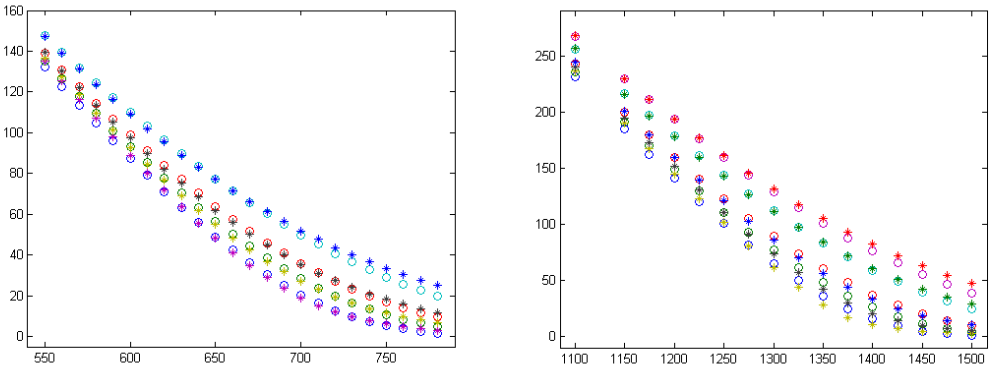



Figure 6.19: Option market prices on 03/03/2008 for RUT (left) and SPX (rings) and prices given by the Variance-Gamma model (stars).

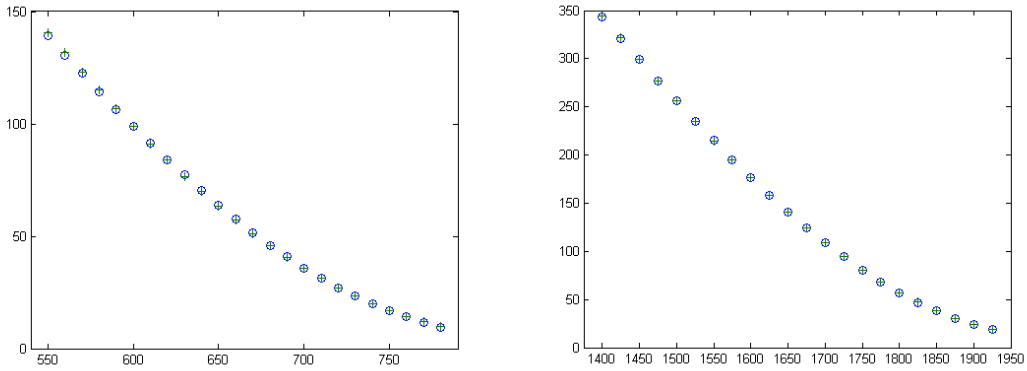



Figure 6.20: Option market prices on 03/03/2008 for RUT June '08 option prices (left) and NDX May 08' option prices (rings) and prices given by the NIG model (left) and the Variance-Gamma model (right) (stars).

calibrate a model for just one maturity (that is to say just one range of strikes) we obtain a perfect calibration (see Figure (6.20)). We chose to calibrate the NIG model to the June '08 RUT options (110 days to maturity) and the VG model to the May '08 NDX options (75 days to maturity). As we can see on the figure, the model prices fit almost perfectly the market prices. In the NIG case, we reach a $1.5e^{-5}$ global square error and $3.0e^{-7}$ in the VG case. The mean error per option is 0.74% in the NIG case and 0.10% for the VG.

Thus, using these four Lévy models, we clearly observe a significant improvement with respect to the Black-Scholes model. We can conclude that the more flexible distributions as JD, DE, NIG and VG are more suitable than the Normal distribution.

# Conclusion

In the first part of the thesis, we introduced some exponential Lévy processes used for option pricing, giving their foremost mathematical properties. Then, we computed option pricing with Fourier Transform methods, based on the knowledge of the characteristic function, performing the FRFT algorithm. Finally we calibrated the model parameters to option market prices with the non-linear least-squares method performing the gradient descent algorithm to find a minimum.

Using an exponential Lévy model we obtained a much better fitting in comparison with the Black-Scholes model (using a single volatility parameter). Indeed, when we calibrate the prices of a set of options over just one maturity, we get high precision approximations. However, it seems almost impossible to calibrate accurately the prices of stock index options of several maturities at the same time. Even starting from various initial parameter sets, the algorithm never reaches a satisfactory minimum (with all the four models, even if worse with jump-diffusion types) and in the mean, we obtain an error between 0.5% and 1.5% for each option price (that error can reach 60% for some strikes and maturities). The framework of exponential Lévy models seems not to be sufficiently flexible to reproduce the term structure of implied volatilities correctly. Surely, a much better method to reach this goal could be the use

of models including both jumps and stochastic volatility. The study of these hybrid models, and particularly the methods used to perform pricing and calibration under these latters, could be a possible extension of this work.

In this thesis, we didn't talk at all about two fundamental topics in financial modeling: pricing of exotic options and hedging. The former is one of the first purposes of option pricing models, especially Lévy models. In fact, the model calibration aim is getting an estimate on the parameters implied in vanilla option prices (available as an exchange quotation) in order to price exotic OTC products with non-vanilla payoffs (as Barrier or Asian options). Very often, pricing of exotic options under Lévy processes is made through Monte Carlo methods, but there also exist numerical methods resolving the associate partial differential equations.
On the other hand, when dealing with options and particularly with exotic ones, hedging is a topic at least as important as pricing if no more. Giving the risk produced by a derivative product, being able to calculate and control this risk is the purpose of a lot of models.
Accordingly, exotic option pricing and hedging could be two possible extensions based on the results of this thesis.

# Appendices

# Appendix A

# Matlab Code

In this appendix, we insert all the MATLAB code we have written to realize the simulations, pricing and calibration.
All the code in this appendix can be downloaded from:

http://ddeville.110.mb/thesis/

Box-Muller algorithm: `BoxM.m`

```
function [A,B] = BoxM(a,b)
U = rand;
V = rand;
E = -2*log(U);
A = sqrt(E)*cos(2*pi*V);
B = sqrt(E)*sin(2*pi*V);
A = a + sqrt(b)*A;
B = a + sqrt(b)*B;
```

Brownian motion with drift $\mu$ and volatility $\sigma$: `Brownian_Motion.m`

```
function Brownian_Motion(mu,sigma)
T = 1;
N = 5000;
h = T/N;  t = (0:h:T);
X(1) = 0;
```

```
for i=1:N
    X(i+1) = X(i) + mu*h + sigma*sqrt(h)*randn;
end
plot(t,X)
```

Sample path under the Black-Scholes model: `Black_Scholes.m`

```
function Black_Scholes(r,sigma,S0,T)
mu = r - 0.5*sigma*sigma;
N = 5000;
h = T/N;  t = (0:h:T);
X(1) = 0;
for i=1:N
    X(i+1) = X(i) + mu*h + sigma*sqrt(h)*randn;
end
S = S0*exp(X);
plot(t,S)
```

Poisson generator: exponential inter-arrival times: `pssrnd1.m`

```
function y = pssrnd1(lambda)
X = 0;
Sum = 0;
flag = 0;
while flag == 0
    E = -log(rand);
    Sum = Sum + E;
    if Sum < lambda
        X = X + 1;
    else
        flag = 1;
    end
end
y = X;
```

Poisson generator: multiplication of uniform random variable: `pssrnd2.m`

```
function y = pssrnd2(lambda)
X = 0;
```

```
Prod = 1;
EE = exp(-lambda);
flag = 0;
while flag == 0
    U = rand;
    Prod = Prod*U;
    if Prod > EE
        X = X + 1;
    else
        flag = 1;
    end
end
y = X;
```

Poisson generator: inversion by sequential search: `pssrnd3.m`

```
function y = pssrnd3(lambda)
X = 0;
Sum = exp(-lambda);
Prod = exp(-lambda);
U = rand;
while U > Sum
    X = X + 1;
    Prod = Prod*(lambda/X);
    Sum = Sum + Prod;
end
y = X;
```

Poisson process with parameter $\lambda$: `PSS_Process.m`

```
function PSS_Process(lambda)
% It requires the function pssrnd1.m
T = 1;
N = 10000;
h = T/N;
t = (0:h:T);
I = zeros(N,1);
X = zeros(N+1,1);  X(1) = 0;
```

```
for i = 1:N
    I(i) = pssrnd1(h*lambda);
    X(i+1) = X(i) + I(i);
end
plot(t,X)
```

Compensated Poisson process with parameter $\lambda$: `CPS_PSS_Process.m`

```
function CPS_PSS_Process(lambda)
% It requires the function pssrnd1.m
T = 1;
N = 5000;
h = T/N;  t = (0:h:T);
I = zeros(N,1);
X = zeros(N+1,1);  X(1) = 0;
for i = 1:N
    I(i) = pssrnd1(h*lambda);
    X(i+1) = X(i) -lambda*h + I(i);
end
plot(t,X)
```

Compound Poisson process with parameter $\lambda$ and Gaussian distribution of jump sizes: `CPD_PSS_Process.m`

```
function CPD_PSS_Process(lambda)
% It requires the function pssrnd1.m
T = 1;
N = 5000;
h = T/N;  t = (0:h:T);
I = zeros(N,1);
X = zeros(N+1,1);  X(1) = 0;
F = zeros(N+1,1);
for i = 1:N
    I(i) = pssrnd1(h*lambda);
    if I(i) == 0;
        F(i) = 0;
    else F(i) = randn;
    end
```

```
    X(i+1) = X(i) + F(i);
end
plot(t,X)
```

Merton Normal Jump-Diffusion with parameters $\mu$, $\sigma$, $\lambda$, $A$ and $B$:

`JD_N_Process.m`

```
function JD_N_Process(mu,sigma,lambda,A,B)
% It requires the function pssrnd1.m
T = 1;
N = 5000;
h = T/N;  t = (0:h:T);
I = zeros(N,1);
X = zeros(N+1,1);  X(1) = 0;
F = zeros(N+1,1);
for i = 1:N
    I(i) = pssrnd1(h*lambda);
    if I(i) == 0;
        F(i) = 0;
    else F(i) = A*I(i) + sqrt(B)*sqrt(I(i))*randn;
    end
    X(i+1) = X(i) + mu*h + sigma*sqrt(h)*randn + F(i);
end
plot(t,X)
```

Kou Double Exponential Jump-Diffusion with parameters $\mu$, $\sigma$, $\lambda$, $p$, $\eta_1$ and $\eta_2$: `JD_DE_Process.m`

```
function JD_DE_Process(mu,sigma,lambda,p,eta1,eta2)
% It requires the functions pssrnd1.m and Gamma1.m
T = 1;
N = 5000;
h = T/N;  t = (0:h:T);
I = zeros(N,1);
X = zeros(N+1,1);  X(1) = 0;
F = zeros(N+1,1);
for i = 1:N
```

```
    I(i) = pssrnd1(h*lambda);
    if I(i) == 0;
        F(i) = 0;
    else K = binornd(I(i),p);
        R1 = Gamma1(K*eta1,eta2);
        R2 = Gamma1((I(i)-K)*eta1,eta2);
        F(i) = R1 - R2;
    end
    X(i+1) = X(i) + mu*h + sigma*sqrt(h)*randn + F(i);
end
plot(t,X)
```

Inverse Gaussian generator from Michael, Schucany and Hass' algorithm (Devroye): `IG1.m`

```
function y = IG1(a,b)
N = randn;
Y = N*N;
X1 = (a/b) + Y/(2*b*b) - (sqrt(4*a*b*Y + Y*Y))/(2*b*b);
U = rand;
if U <= (a/(a+X1*b))
    X = X1;
else X = (a*a)/(b*b*X1);
end
y = X;
```

Inverse Gaussian generator from Michael, Schucany and Hass' algorithm (Glasserman): `IG2.m`

```
function y = IG2(a,b)
C = 1/b;
D = C*a;
D = D*D;
Norm = randn;
V = Norm*Norm;
eps = C*V;
Y = C*(a+(eps/2) + sqrt(eps*(a+(eps/4))));
```

```
p = a/(a+b*Y);
U = rand;
if U > p
    Y = D/Y;
end
y = Y;
```

Sample path of an IG process with parameters $a$ and $b$: `IG_Process.m`

```
function IG_process(a,b)
% It requires the function IG2.m
T = 1;
N = 1000;
h = T/N;  t = (0:h:T);
g = zeros(N,1);
G = zeros(N+1,1);  G(1) = 0;
for i = 1:N
    g(i) = IG2(a*h,b);
    G(i+1) = G(i) + g(i);
end
plot(t,G)
```

Sample path of a NIG process with parameters $\alpha$, $\beta$ and $\delta$ by Brownian subordination: `NIG_Sub_Brown.m`

```
function NIG_Sub_Brown(alpha,beta,delta)
% It requires the function IG2.m
a = 1;
b = delta*sqrt(alpha*alpha - beta*beta);
T = 1;
N = 1000;
h = T/N;  t = (0:h:T);
I = zeros(N,1);
X = zeros(N+1,1);  X(1) = 0;
for i = 1:N
    I(i) = IG2(a*h,b);
    X(i+1) = X(i) + beta*delta*delta*I(i)
     + delta*sqrt(I(i))*randn;
```

```
end
plot(t,X)
```

Gamma random numbers generator with Ahrens-Dieter and Fishman's algorithms: `Gamma1.m`

```
function y = Gamma1(a,b)
if a == 0;
    answ = 0;
elseif a <= 1
    answ = gamma1(a);
else
    answ = gamma2(a);
end
y = answ/b;

% Ahrens-Dieter's Gamma generator (if a<=1)
function y = gamma1(a)
e = exp(1);
c = (a+e)/e;
flag = 0;
while flag == 0
    U1 = rand;
    U2 = rand;
    Y = c*U1;
    if Y<=1
        Z = Y^(1/a);
        if U2<exp(-Z)
            flag = 1;
        end
    else Z = -log((c-Y)/a);
        if U2<=Z^(a-1)
            flag = 1;
        end
    end
end
y = Z;
```

```
% Fishman (Cheng and Feast)'s Gamma generator (if a>1)
function y = gamma2(a)
a2 = a-1;
c = (a-(1/(6*a)))/a2;
m = 2/a2;
d = m+2;
flag = 0;
while flag == 0
    U1 = rand;
    U2 = rand;
    V = c*U2/U1;
    if m*U1-d+V+(1/V)<=0
        flag = 1;
    elseif m*log(U1)-log(V)+V-1<=0
        flag = 1;
    end
end
y = a2*V;
```

Gamma random numbers generator with Johnk and Best's algorithms:

Gamma2.m

```
function y = Gamma2(a,b)
if a == 0;
    answ = 0;
elseif a <= 1
    answ = gamma1(a);
else
    answ = gamma2(a);
end
y = answ/b;

% Johnk's Gamma generator (if a<=1)
function y = gamma1(a)
X = 0;
Y = 0;
while X + Y <= 1
U = rand;
```

```
V = rand;
X = U^(1/a);
    if a == 1
        Y = 1;
    else Y = V^(1/(1-a));
    end
end
E = (-log(rand));
y = (X*E)/(X+Y);

% Best's Gamma generator (if a>1)
function y = gamma2(a)
d = a-1;
c = 3*a - 3/4;
ifl = 0;
while(ifl == 0)
    U = rand;
    V = rand;
    W = U*(1-U);
    Y = sqrt(c/W)*(U-0.5);
    X = d + Y;
    Z = 64*W^3*V^3;
    if(log(Z)<=2*(d*log(X/d)-Y))
        if(X>0)
            ifl = 1;
        end
    end
end
y = X;
```

Gamma random numbers generator with Berman and Best's algorithms:

Gamma3.m

```
function y = Gamma3(a,b)
if a == 0;
    answ = 0;
elseif a <= 1
    answ = gamma1(a);
```

```
else
    answ = gamma2(a);
end
y = answ/b;

% Berman's Gamma generator (if a<=1)
function y = gamma1(a)
X = 0;
Y = 0;
while X + Y <= 1
U = rand;
V = rand;
X = U^(1/a);
    if a == 1
        Y = 1;
    else Y = V^(1/(1-a));
    end
end
U = rand;
V = rand;
y = -X*log(U*V);

% Best's Gamma generator (if a>1)
function y = gamma2(a)
d = a-1;
c = 3*a - 3/4;
ifl = 0;
while(ifl == 0)
    U = rand;
    V = rand;
    W = U*(1-U);
    Y = sqrt(c/W)*(U-0.5);
    X = d + Y;
    Z = 64*W^3*V^3;
    if(log(Z)<=2*(d*log(X/d)-Y))
        if(X>0)
            ifl = 1;
        end
```

```
    end
end
y = X;
```

Sample path of a Gamma process with parameters $a$ and $b$: `Gamma_Process.m`

```
function Gamma_process(a,b)
% It requires the function Gamma1.m
T = 1;
N = 1000;
h = T/N;  t = (0:h:T);
g = zeros(N+1,1);
G = zeros(N+1,1);  G(1) = 0;
for i = 1:N
    g(i) = Gamma1(a*h,b);
    G(i+1) = G(i) + g(i);
end
plot(t,G)
```

Sample path of a Variance Gamma process with parameters $\sigma$, $\nu$ and $\theta$ by Brownian Subordination: `VG_Sub_Brown.m`

```
function VG_Sub_Brown(sigma,nu,theta)
% It requires the function Gamma1.m
a = 1/nu;
b = 1/nu;
T = 1;
N = 1000;
h = T/N;  t = (0:h:T);
X = zeros(N+1,1);
I = zeros(N+1,1);
for i = 1:N
    I(i) = Gamma1(a*h,b);
    X(i+1) = X(i) + theta*I(i) + sigma*sqrt(I(i))*randn;
end
plot(t,X)
```

Sample path of a Variance Gamma process with parameters $C$, $G$ and $M$ as the difference of two Gamma processes: `VG_Diff_Gamma.m`

```
function VG_Diff_Gamma(C,G,M)
% It requires the function Gamma1.m
a1 = C; b1 = M;
a2 = C; b2 = G;
T = 1;
N = 1000;
h = T/N;  t = (0:h:T);
G1 = zeros(N+1,1); G2 = zeros(N+1,1);
VG = zeros(N+1,1);
for i = 1:N
    G1(i+1) = G1(i) + Gamma1(a1*h,b1);
    G2(i+1) = G2(i) + Gamma1(a2*h,b2);
    VG(i+1) = G1(i) - G2(i);
end
plot(t,VG)
```

Function to convert the $\sigma, \nu, \theta$ parameters of the VG process into the $C, G, M$ parameters: `Change_VG.m`

```
function Change_VG(sigma,nu,theta)
C = 1/nu;
G = (sqrt(0.25*theta*theta*nu*nu+0.5*sigma*sigma*nu)
    -0.5*theta*nu)^(-1);
M = (sqrt(0.25*theta*theta*nu*nu+0.5*sigma*sigma*nu)
    +0.5*theta*nu)^(-1);
fprintf('\t%+6.4f \t\n',C);
fprintf('\t%+6.4f \t\n',G);
fprintf('\t%+6.4f \t\n',M);
```

The Merton Series Expansion: `MertonSerie.m`

```
function Value = MertonSerie(S, K, r, T, sigma, lambda,
                 jumpa, jumpb, MaxIter)
Value = 0;
lambdabis = lambda*exp(jumpa + 0.5*jumpb^2);
for i = 0:MaxIter
    Vi = sqrt(sigma^2 + (i*jumpb^2)/T);
    ri = r + (i/T)*(jumpa+0.5*jumpb^2)-lambda*(exp(jumpa
```

```
       + 0.5*jumpb^2)-1);
    Value = Value + (exp(-lambdabis * T) * (lambdabis *
          T) ^ i / factorial(i)) * bs(S, K, ri, Vi, T);
end

function call = bs(S, K, r, sigma, t)
d1 = ( log(S./K)+( r + .5*sigma.^2 ).*t ) ./ sigma./sqrt(t);
d2 = d1 - sigma.*sqrt(t); n1 = normcdf(d1); n2 = normcdf(d2);
call = S.*n1 - K.*n2.*exp(-r.*t);
```

Functions that define the characteristic functions and $\psi$ functions (see Carr and Madan, 1999) for Black-Scholes model, Merton Jump-Diffusion model, Kou Double-Exponential Jump-Diffusion model, Variance Gamma and Normal Inverse Gaussian.

Black-Scholes model: `BS_CF.m` and `Psi_BS_CF.m`

```
function y = BS_CF(u, sigma, r, t)
% Computes the characteristic function for
% the Black-Scholes model
drift = r - 0.5*sigma^2;
y = exp(i*drift*t*u - 0.5*sigma^2*u.^2*t);

function y = Psi_BS_CF(u, sigma, r, t, a)
% computes the Psi function (modified call, Carr-Madan, 1999)
y1 = BS_CF(u - i*(a+1), sigma, r, t);
y2 = a^2 + a - u.^2 + i*(2*a+1)*u;
y  = exp(-r*t) * y1 ./ y2;
```

Merton Jump-Diffusion model: `JD_CF.m` and `Psi_JD_CF.m`

```
function y = JD_CF(u, sigma, lambda, jumpa, jumpb, r, t)
% Computes the characteristic function for the Jump-Diffusion model
drift = r - 0.5*sigma^2 - lambda*(exp(jumpa+0.5*jumpb^2)-1);
y = exp(i*drift*t*u - 0.5*sigma^2*u.^2*t + t*lambda*(exp(i *
    jumpa*u - 0.5*jumpb^2*u.^2)-1));
```

```
function y = Psi_JD_CF(u, sigma, lambda, jumpa, jumpb, r, t, a)
% computes the Psi function (modified call, Carr-Madan, 1999)
y1 = JD_CF(u - i*(a+1), sigma, lambda, jumpa, jumpb, r, t);
y2 = a^2 + a - u.^2 + i*(2*a+1)*u;
y  = exp(-r*t) * y1 ./ y2;
```

Kou Double-Exponential Jump-Diffusion: `DE_CF.m` and `Psi_DE_CF.m`

```
function y = DE_CF(u, sigma, lambda, prob, eta1, eta2, r, t)
% Computes the characteristic function for the
% Jump-Diffusion model
drift = r - 0.5*sigma^2 - lambda*(((prob*eta1)/(eta1+1))+
        ((1-prob)*eta2/(eta2+1))-1);
y = exp(i*u*drift*t - 0.5*sigma^2*u.^2*t + t*lambda*((prob*
    eta1)./(eta1+u*i) + ((1-prob)*eta2)./(eta2+u*i) - 1 ));

function y = Psi_DE_CF(u,sigma,lambda,prob,eta1,eta2,r,t,a)
% computes the Psi function (modified call, Carr-Madan, 1999)
y1 = DE_CF(u - i*(a+1), sigma, lambda, prob, eta1, eta2, r, t);
y2 = a^2 + a - u.^2 + i*(2*a+1)*u;
y  = exp(-r*t) * y1 ./ y2;
```

Variance Gamma: `VG_CF.m` and `Psi_VG_CF.m`

```
function y = VG_CF(u, sigma, theta, nu, r, t)
% Computes the characteristic function for the
% Variance-Gamma model
drift = r + log(1-theta*nu-0.5*sigma^2*nu)/nu;
y = exp(i*drift*t*u) .* ((1-i*nu*theta*u+0.5*nu
    *sigma^2*u.^2).^(-t/nu));

function y = Psi_VG_CF(u, sigma, theta, nu, r, t, a)
% computes the Psi function (modified call, Carr-Madan, 1999)
y1 = VG_CF(u - i*(a+1), sigma, theta, nu, r, t);
y2 = a^2 + a - u.^2 + i*(2*a+1)*u;
y  = exp(-r*t) * y1 ./ y2;
```

Normal Inverse Gaussian: `NIG_CF.m` and `Psi_NIG_CF.m`

```
function y = NIG_CF(u, delta, alph, beta, r, t)
% Computes the characteristic function for the NIG model
drift = r + delta*(sqrt(alph^2-(beta+1)^2)-
sqrt(alph^2-beta^2));
y = exp(i*u*drift*t - delta*t*(sqrt(alph^2-(beta+i*u).^2)
- sqrt(alph^2-beta^2)));

function y = Psi_NIG_CF(u, delta, alph, beta, r, t, a)
% computes the Psi function (modified call, Carr-Madan, 1999)
y1 = NIG_CF(u - i*(a+1), delta, alph, beta, r, t);
y2 = a^2 + a - u.^2 + i*(2*a+1)*u;
y  = exp(-r*t) * y1 ./ y2;
```

Function that computes the FFT procedure to find an option price with the jump-diffusion model: `JD_FFT.m`

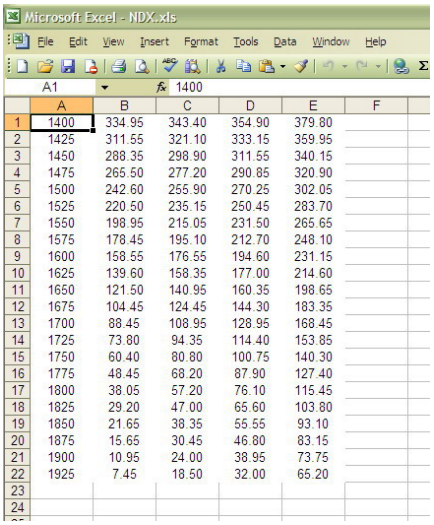
```
function JD_FFT
% It requires the function JD_CF and Psi_JD_CF
% Model parameters:
sigma = 0.25;   % Volatility
r = 0.05;       % Risk-free rate
t = 0.25;       % Time to maturity
lambda = 0.1;   % Jump/year
jumpa = -0.05;  % Mean jump
jumpb = 0.3;    % Jump volatility
S = 1;          % Initial stock price
eprice = 1;     % Strike price
% Model parameter set
params = [sigma, lambda, jumpa, jumpb, r, t];

% Definition of the integration grid:
N = 4096;       % Number of FFT points
a = 1024;       % Upper integration bound (0,+a)
alpha = 2.5;    % Dampening factor (hint: 4)

% Outputs
% K : Set of strikes
% Y : Set of option prices
```

```
[K, Y] = JD_FFT_PRICES(S,params, N, a, alpha);

% We compute the option price for the given strike price
% by interpolation:
j=1;
while K(j)<eprice
   j=j+1;
end
option = ((Y(j)-Y(j-1))*(eprice-K(j-1)))/(K(j)-K(j-1))+Y(j-1);
fprintf('\n\nFFT Price:\n'); fprintf('%+6.4f\n',option);


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% Functions that compute the program %%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [K, Y] = JD_FFT_PRICES(S, params, N, a, alpha)
% Parameters for the CF grid (0,a)
delta = a/N;              % Integration grid spacing
lamb = (2*pi)/(N*delta); % Log-strikes grid spacing
x0 = -N*lamb/2;           % 1st pt of the log-strikes grid
% Creation of the grid
u = (0:N-1) * delta;      % Support for integration
x = x0 + (0:N-1) * lamb; % Support for log-strikes
% Application of the rules on the grid
h = Psi_CF(u, params, alpha);   % Psi function values
                                % on the integration grid
h2 = delta * exp(-i*x0*u) .* h; % Function to integrate
TR = [0.5 ones(1,N-2) 0.5];     % Trapezoid rule
h3 = h2 .* TR;                  % Apply rule
g = fft(h3);                    % Application of the FFT
g2 = real(exp(-alpha*x)/pi.*g); % Normalized prices
% Compute output values
K = S * exp(x);
Y = S * g2;
```

Function that computes the FRFT procedure to find an option price with the Variance-Gamma model: `VG_FRFT.m`

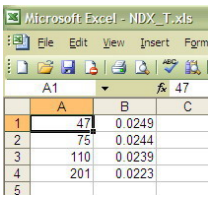
```
function VG_FRFT
It requires the functions CF_VG and Psi_VG_CF
% Model parameters:
sigma = 0.30;   % Volatility
r = 0.05;       % Risk-free rate
t = 0.25;       % Time to maturity
theta = -0.20;
nu = 0.20;
S = 1;          % Initial stock price
eprice = 1;     % Strike price
% Model parameter set
params = [sigma, theta, nu, r, t];

% Definition of the integration grid:
N = 128;        % Number of FFT points
a = 140;        % Upper integration bound (0,+a)
b = 0.3;        % Bounds for log-prices (-b,+b)
alpha = 2.5;    % Dampening factor (hint: 4)

% Outputs
% K : Set of strikes
% Y : Set of option prices
[K, Y] = VG_FRFT_PRICES(S,params, N, a, b, alpha);

% We compute the option price for the given strike price by
% interpolation:
j=1;
while K(j)<eprice
   j=j+1;
end
option = ((Y(j)-Y(j-1))*(eprice-K(j-1)))/(K(j)-K(j-1))+Y(j-1);
fprintf('\n\nFRFT Price:\n'); fprintf('%+6.4f\n',option);
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% Functions that compute the program %%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [K, Y] = VG_FRFT_PRICES(S, params, N, a, b, alpha)
% Parameters for the CF grid (0,a)
delta = a/N;            % Integration grid spacing
lamb = 2*b/N;           % Log-strikes grid spacing
x0 = -b;                % 1st pt of the log-strikes grid
eps = 0.5*lamb*delta/pi;    % Fractional parameter
% Creation of the grid
u = (0:N-1) * delta;        % Support for integration
x = x0 + (0:N-1) * lamb;    % Support for log-strikes
% Application of the rules on the grid
h = Psi_CF(u, params, alpha);   % Psi function values
                                % on the integration grid
h2 = delta * exp(-i*x0*u) .* h; % Function to integrate
TR = [0.5 ones(1,N-2) 0.5];     % Trapezoid rule
h3 = h2 .* TR;                  % Apply rule
g = FRFT(h3, eps);              % Application of the FRFT routine
g2 = real(exp(-alpha*x)/pi.*g); % Normalized prices
% Compute output values
K = S * exp(x);
Y = S * g2;

function f = FRFT(x, a)
% Performs the Fractional FFT on a vector 'x' with
% fractional parameter 'a'
% From Chourdakis
m = size(x,2);
% Auxilliary vectors
y = zeros(1,2*m);
y(1,1:m) = x .* exp( - pi*i*a * (0:(m-1)).^2 );
z = zeros(1,2*m);
z(1,1:m) = exp( pi*i*a * (0:(m-1)).^2 );
z(1,m+1:2*m) = exp( pi*i*a * ( (m:(2*m-1)) - 2*m).^2 );
% The three (I)FFTs
fy = fft(y); fz = fft(z); fyz = fy .* fz; ifyz = ifft(fyz);
```




| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | 1400 | 334.95 | 343.40 | 354.90 | 379.80 | |
| 2 | 1425 | 311.55 | 321.10 | 333.15 | 359.95 | |
| 3 | 1450 | 288.35 | 298.90 | 311.55 | 340.15 | |
| 4 | 1475 | 265.50 | 277.20 | 290.85 | 320.90 | |
| 5 | 1500 | 242.60 | 255.90 | 270.25 | 302.05 | |
| 6 | 1525 | 220.50 | 235.15 | 250.45 | 283.70 | |
| 7 | 1550 | 198.95 | 215.05 | 231.50 | 265.65 | |
| 8 | 1575 | 178.45 | 195.10 | 212.70 | 248.10 | |
| 9 | 1600 | 158.55 | 176.55 | 194.60 | 231.15 | |
| 10 | 1625 | 139.60 | 158.35 | 177.00 | 214.60 | |
| 11 | 1650 | 121.50 | 140.95 | 160.35 | 198.65 | |
| 12 | 1675 | 104.45 | 124.45 | 144.30 | 183.35 | |
| 13 | 1700 | 88.45 | 108.95 | 128.95 | 168.45 | |
| 14 | 1725 | 73.80 | 94.35 | 114.40 | 153.85 | |
| 15 | 1750 | 60.40 | 80.80 | 100.75 | 140.30 | |
| 16 | 1775 | 48.45 | 68.20 | 87.90 | 127.40 | |
| 17 | 1800 | 38.05 | 57.20 | 76.10 | 115.45 | |
| 18 | 1825 | 29.20 | 47.00 | 65.60 | 103.80 | |
| 19 | 1850 | 21.65 | 38.35 | 55.55 | 93.10 | |
| 20 | 1875 | 15.65 | 30.45 | 46.80 | 83.15 | |
| 21 | 1900 | 10.95 | 24.00 | 38.95 | 73.75 | |
| 22 | 1925 | 7.45 | 18.50 | 32.00 | 65.20 | |
| 23 | | | | | | |
| 24 | | | | | | |

Figure A.1: NDX.xls file containing the NDX strike and market prices.




| | A | B | C |
|---|---|---|---|
| 1 | 47 | 0.0249 | |
| 2 | 75 | 0.0244 | |
| 3 | 110 | 0.0239 | |
| 4 | 201 | 0.0223 | |
| 5 | | | |

Figure A.2: NDX_T.xls file containing NTX maturities and discount factors.

```
f = exp( - pi*i*a * (0:(m-1)).^2 ) .* ifyz(1,1:m);
```

Function that first loads market prices (in this case NDX) from a *.xls* file (see Figures (A.1) and (A.2)) and calibrates the model parameters (in this case NIG) to the market prices performing a non-linear least-square procedure with the gradient descent algorithm: `cal_NIG_NDX.m`

```
function cal_NIG_NDX(alph,beta,delta)
% It requires the function NIG_FRFT.m
% The required parameters in input are
% the initially guessed parameters.
par = [alph,beta,delta];
```

```
par_new = par;

Spot = 1725.52;             % Spot NDX price
OP = xlsread('NDX.xls');    % We load NDX option prices
TD = xlsread('NDX_T.xls');  % We load NDX option maturities

% We ordinate and organize the inputs:
[no,mo] = size(OP);
strike = OP(:,1)/Spot;      % Strike prices vector
C_mark = OP(:,2:mo)/Spot;   % Option prices matrix
t = (1/360)*TD(:,1);        % Maturites vector
r = TD(:,2);                % Interest-rate vector
S=1;    % We fix S=1 and get everyting in function of S.

% Indexation:
error=zeros(1,100000);
error(1) = Psi(par,C_mark,S,strike,r,t);

itmax = 100000;     % Iteration max
tol = 1e-4;         % Error tolerance

% Steps:
hstep = 1e-2;   % Gradient descent step
hstep1 = 1e-5;  % Finite-Differences step

fprintf('Initial values of parameters : \n')
disp(par)
fprintf('Press any key \n')
pause

k = 1; ifl = 0;
while(ifl == 0)
    k = k+1;
    if(k > itmax)
        ifl = 1;
    end
    if(error(k-1) < tol)
        fprintf('Calibrated parameters : \n')
```

```
        disp(par)
        ifl=1;
    else
        for j = 1:3
            par_aux = par;
            par_aux(j)=par_aux(j) + hstep1;
            error_aux = Psi(par_aux,C_mark,S,strike,r,t);
            grad(j)=(error_aux - error(k-1))/hstep1;
        end
        for j = 1:3
            par_new(j) = par(j) - hstep*grad(j);
        end
        error(k) = Psi(par_new,C_mark,S,strike,r,t);
    end
    error(k);
    par = par_new;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% Functions that Compute the Program %%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Function that computes the error squares:
function y = Psi(par,C_mark,S,strike,r,t)
C_actual = NIGprice(par,S,strike,r,t);  % Model prices
C_market = C_mark;                      % Market prices
y = (sum(sum(C_actual - C_market).^2))  % Function to minimize

% Calculation of Option prices with FRFT:
function y = NIGprice(par,S,strike,r,t)
alph = par(1); beta = par(2);
delta = par(3);
for i=1:4
    for j=1:22
       dc(i,j)=NIG_FRFT(S,strike(j),r(i),t(i),delta,alph,beta);
    end
end
y = dc';
```

Function that plots the market and model (in this case Variance Gamma) given the model parameters: Plot_VG_NDX.m

```
function Plot_VG_NDX(sigma,theta,nu)
% It requires the function VG_FRFT.m
% The required parameters in input are
% those found after calibration.

S = 1725.52;                % Spot Price
OP = xlsread('NDX.xls');    % We load NDX option prices
TD = xlsread('NDX_T.xls');  % We load NDX option maturities

% We ordinate and organize the inputs:
[np,mp] = size(OP);
strike = OP(:,1);           % Strike prices vector
C_mark = OP(:,2:mp);        % Option prices matrix
[no,mo] = size(C_mark);
t = (1/365)*TD(:,1);        % Maturites vector
r = TD(:,2);                % Interest-rate vector

for i=1:mo
    for j=1:no
       option(i,j)=VG_FRFT(S,strike(j),r(i),t(i),sigma,theta,nu);
    end
end
option = option';

% We plot both market and model prices
plot(strike,C_mark,'o',strike,option,'*')
```

Function that calibrates the NIG model parameters to option market prices on RUT with maturities June, 08, performing a non-linear least-square procedure with the gradient descent algorithm: cal_NIG_RUT_110.m

```
function cal_NIG_RUT(alph,beta,delta)
% It requires the function NIG_FRFT.m
```

```
% alph = -18.601;
% beta = -12.247;
% delta = 0.835;

% We load all the option data:
S = 680.73;
strike = (1/S)*[550 560 570 580 590 600 610 620 630 640 ...
... 650 660 670 680 690 700 710 720 730 740 750 760 770 780];
C_mark = (1/S)*[139.05 130.65 122.45 114.40 106.50 98.90 ...
... 91.45 84.15 77.25 70.45 63.95 57.55 51.65 46.05 40.90 ...
35.80 31.25 27.10 23.25 19.75 16.75 14.05 11.55 9.55];
strike = strike';
C_mark = C_mark';
t = 110/365;
r = 0.0082;
S=1;

par = [alph,beta,delta];
par_new = par;

error=zeros(1,100000);
error(1) = Psi(par,C_mark,S,strike,r,t);

itmax = 100000;     % max iteration
tol = 1.5e-5;       % error tolerance

hstep = 2;
hstep1 = 1e-5;

fprintf('Initial values of parameters : \n')
disp(par)
fprintf('Press any key \n')
pause

k = 1; ifl = 0;
while(ifl == 0)
    k = k+1;
    if(k > itmax)
```

```
        ifl = 1;
    end
    if(error(k-1) < tol)
        fprintf('Calibrated parameters : \n')
        disp(par)
        ifl=1;
    else
        for j = 1:3
            par_aux = par;
            par_aux(j)=par_aux(j) + hstep1;
            error_aux = Psi(par_aux,C_mark,S,strike,r,t);
            grad(j)=(error_aux - error(k-1))/hstep1;
        end
        for j = 1:3
            par_new(j) = par(j) - hstep*grad(j);
        end
        error(k) = Psi(par_new,C_mark,S,strike,r,t);
    end
    error(k);
    par = par_new;
end


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% Functions that compute the program %%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function y = Psi(par,C_mark,S,strike,r,t)
C_actual = NIGprice(par,S,strike,r,t);
C_market = C_mark;
y = sum((C_actual - C_market).^2)

function y = NIGprice(par,S,strike,r,t)
alph = par(1); beta = par(2);
delta = par(3);
for i=1:24
        dc(i) = NIG_FRFT(S,strike(i),r,t,delta,alph,beta);
end
```

```
y = dc';
```

Function that plots the market (NDX) and model (in this case Variance Gamma) for just one maturity (May '08) given the model parameters and return the mean percentuale error: `Plot_VG_NDX_75.m`

```
function Plot_VG_NDX_75(sigma,theta,nu)

% sigma = 0.23065153252436;
% theta = -0.70197610213548;
% nu = 0.08440167872024;

% We load all the option data:
S = 1725.52;
strike = [1400 1425 1450 1475 1500 1525 1550 1575...
... 1600 1625 1650 1675 1700 1725 1750 1775 1800...
... 1825 1850 1875 1900 1925];
C_mark = [343.40 321.10 298.90 277.20 255.90...
... 235.15 215.05 195.10 176.55 158.35 140.95...
... 124.45 108.95 94.35 80.80 68.20 57.20 47.00...
... 38.35 30.45 24.00 18.50];
t = 75/365;
r = 0.0244;

for i=1:22
        option(i) = VG_FRFT(S,strike(i),r,t,sigma,theta,nu);
end
option = option';


x = (sum(C_mark));
y = (sum(abs(C_mark' - option))/x)*100;
disp('The mean percentuale error per option is: ')
disp(y)

plot(strike,C_mark,'o',strike,option,'+')
axis([1375 1950 0 350]);
```

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